arXiv:0804.1515v2 [math-ph] 30 Mar 2009

On properties of the space of quantum states and their application to construction of entanglement monotones

M.E.Shirokov*
Steklov Mathematical Institute, Moscow, Russia

Contents

*Work is partially supported by the program "Modern problems of theoretical mathematics" of Russian Academy of Sciences, by RFBR grant 06-01-00164-a.

1 Introduction

In study of finite dimensional quantum systems and channels such notions of the convex analysis as the convex hull and the convex closure (called also the convex envelope) of a function defined on the set of quantum states, the convex roof of a function defined on the set of pure quantum states (introduced in [30] as a special convex extension of this function to the set of all quantum states) are widely used. The last notion plays the basic role in construction of entanglement monotones – functions on the set of states of a composite quantum system characterizing entanglement of these states [19, 32].

The main problem of using these functional constructions in the infinite dimensional case consists in the fact that in this case it is necessary to apply them to functions with singular properties such as discontinuity and unboundedness (including possibility of the infinite values). For instance, the von Neumann entropy – one of the main characteristics of quantum states – is a continuous and bounded function in finite dimensions, but it takes the value $+\infty$ on the dense subset of the set of states of infinite dimensional quantum system. The other problems are noncompactness of the set of quantum states and nonexistence of inner points of this set (considered as a subset of the Banach space of trace class operators). All these features lead to very "unnatural" behavior of the above functional constructions and to breaking of validity of several "elementary" results (for example, the well known Jensen's inequality may not hold for a measurable convex function). So, the special analysis is required to overcome these problems. The main instruments of this analysis are the following two properties of the set of quantum states as a convex topological space:

- weak compactness of the set of measures, whose barycenters form a compact set;
- openness of the barycenter map (in the weak topology),

proved in [11] and in [27] respectively and described in detail in section 2.

These properties reveal the special relations between the topology and the convex structure of the set of quantum states. It is possible to show that their validity for *arbitrary* convex complete separable metric space leads to some nontrivial results concerning properties of functions on this set. Some of these results are considered in [20, 28]. In section 3 we present several

new consequences of the above two properties of the set of quantum states and construct some counterexamples showing their importance. These counterexamples also show the special role of the space of trace class operators (the Shatten class of order $p = 1$) in the family of the Shatten classes of order $p \geq 1$. Continuity of the operations of the convex closure and of the convex roof with respect to monotonous pointwise convergence on the class of lower semicontinuous lower bounded functions is proved.

The sufficient condition of continuity and of coincidence of the restrictions of the convex hulls (roofs) of a given function to the set of states with bounded values of a given lower semicontinuous nonnegative affine function (generalized mean energy) is obtained in section 4.

The two possible infinite dimensional generalizations of the convex roof construction of entanglement monotones are considered and their properties are discussed in section 5. It is shown that the "natural" discrete generalization may be "false" in the sense that the functions constructed by using this method may have not the main property of entanglement monotone (even when the generating function is bounded and lower semicontinuous), while the continuous generalization produces "true" entanglement monotone under the weak requirements on the generating function. So, the last method is considered as a generalized convex roof construction. The properties of entanglement monotones (including generalized monotonicity and different forms of continuity) produced by this construction are obtained and some examples are presented.

The convex roof construction provides infinite dimensional generalization of the notion of the Entanglement of Formation (EoF)– one of the basic measures of entanglement in finite dimensional composite systems [2]. The properties of this generalization of the EoF are considered and its relations to the another possible definition of the EoF proposed in [6] are discussed in section 6.

2 Preliminaries

2.1 Notations

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{B}_h(\mathcal{H})$ – respectively the Banach spaces all bounded operators and of bounded Hermitian operators in $\mathcal{H}$, containing the cone $\mathfrak{B}_+(\mathcal{H})$ of all positive operators, $\mathfrak{T}(\mathcal{H})$ and $\mathfrak{T}_h(\mathcal{H})$ –

respectively the Banach spaces of all trace class operators and of trace class Hermitian operators with the trace norm $\|\cdot\|_1 = \mathrm{Tr}|\cdot|$. Let

$$\mathfrak{T}_1(\mathcal{H}) = \{A \in \mathfrak{T}(\mathcal{H}) \,|\, A \geq 0, \mathrm{Tr}A \leq 1\} \ \text{ and } \ \mathfrak{S}(\mathcal{H}) = \{A \in \mathfrak{T}_1(\mathcal{H}) \,|\, \mathrm{Tr}A = 1\}$$

be the closed convex subsets of $\mathfrak{T}(\mathcal{H})$, which are complete separable metric spaces with the metric defined by the trace norm. Operators in $\mathfrak{S}(\mathcal{H})$ are called density operators or states since each density operator uniquely defines a normal state on the algebra $\mathfrak{B}(\mathcal{H})$.

We will denote by $\mathrm{co}\mathcal{A}$ ($\overline{\mathrm{co}}\mathcal{A}$) the convex hull (closure) of a set $\mathcal{A}$. We will denote by $\mathrm{extr}\mathcal{A}$ the set of all extreme points of a convex set $\mathcal{A}$ [14],[22].

Let $\mathcal{P}(\mathcal{A})$ be the set of all Borel probability measures on complete separable metric space $\mathcal{A}$ endowed with the topology of weak convergence [3],[18]. This set can be considered as a complete separable metric space as well [18]. The subset of $\mathcal{P}(\mathcal{A})$ consisting of measures with finite support will be denoted by $\mathcal{P}^{\mathrm{f}}(\mathcal{A})$. In what follows we will also use the abbreviations $\mathcal{P} = \mathcal{P}(\mathfrak{S}(\mathcal{H}))$, $\widehat{\mathcal{P}} = \mathcal{P}(\mathrm{extr}\mathfrak{S}(\mathcal{H}))$ and $\widehat{\mathcal{P}}(\mathcal{A}) = \mathcal{P}(\mathrm{extr}\mathcal{A})$ for arbitrary convex set $\mathcal{A}$.

The *barycenter* of the measure $\mu \in \mathcal{P}(\mathcal{A})$ is the state in $\overline{\mathrm{co}}\mathcal{A}$ defined by the Bochner integral

$$\bar{\rho}(\mu) = \int_{\mathcal{A}} \sigma\mu(d\sigma).$$

For arbitrary subset $\mathcal{B}$ of $\overline{\mathrm{co}}\mathcal{A}$ let $\mathcal{P}_{\mathcal{B}}(\mathcal{A})$ be the subset of $\mathcal{P}(\mathcal{A})$ consisting of all measures with the barycenter in $\mathcal{B}$.

A collection of states $\{\rho_i\}$ with corresponding probability distribution $\{\pi_i\}$ is conventionally called *ensemble* and is denoted $\{\pi_i, \rho_i\}$. In this paper we will consider ensemble of states as a particular case of probability measure, so that the notation $\{\pi_i, \rho_i\} \in \mathcal{P}_{\{\rho\}}$ means that $\rho = \sum_i \pi_i \rho_i$.

Following [11] an arbitrary positive unbounded operator H in separable Hilbert space $\mathcal{H}$ with discrete spectrum of finite multiplicity will be called $\mathfrak{H}$-operator.

From mathematical point of view the set $\mathfrak{S}(\mathcal{H})$ of quantum states is a noncompact convex complete separable metric space, having the following two properties:

A) for arbitrary compact subset $\mathcal{A} \subset \mathfrak{S}(\mathcal{H})$ the subset $\mathcal{P}_{\mathcal{A}}$ is compact [11];

B) the barycenter map $\mathcal{P} \ni \mu \mapsto \bar{\rho}(\mu) \in \mathfrak{S}(\mathcal{H})$ is an open surjection [27].

2.2 Property A

Property A provides generalization to the case of $\mathfrak{S}(\mathcal{H})$ of some well known results concerning compact convex sets (see lemma 1 in [12] or propositions 1 and 6 below) and hence it may be considered as a kind of "weak" compactness. In fact this property is not purely topological (in contrast to compactness), but it reveals the special relation between the topology and the convex structure[1] of the set $\mathfrak{S}(\mathcal{H})$. The analog of property A for arbitrary closed (generally nonconvex) bounded subset of separable Banach space[2] is considered in [28], where it is called the μ-compactness property. It turns out that μ-compact convex sets inherit some important properties of compact convex sets such as the Choquet theorem of barycentric representation, lower semicontinuity of the convex hull of any continuous bounded function, etc. It is possible to prove μ-compactness of the following important convex sets (see [20]):

1) bounded part of the positive cone of the Banach space $\mathfrak{T}(\mathcal{H})$ of trace class operators;

2) variation bounded set of positive Borel measures on *any* complete separable metric space endowed with the weak topology;

3) norm bounded set of all positive linear operators in the Banach space $\mathfrak{T}(\mathcal{H})$ endowed with the strong operator topology.

It is interesting to note that the Banach space of trace class operators (the Shatten class of order $p = 1$) can not be replaced in 1) by the Shatten class of order $p > 1$. This follows, in particular, from comparison of proposition 3 in [28] with the example in remark 1 in section 3. Moreover, it can be shown that the set $\mathfrak{T}_1(\mathcal{H})$ loses[3] the μ-compactness property being endowed with the $\|\cdot\|_p$-norm topology with $p > 1$ and that in the Shatten class of order $p = 2$ – the Hilbert space of Hilbert-Schmidt operators – there exists no μ-compact set which is not compact [20].

[1] The convex structure is involved in the construction of the barycenter of a probability measure.

[2] It is possible to consider convex complete metrizable bounded subsets of any separable locally convex topological space [20].

[3] This shows that the μ-compactness property is not saved in general under passing to weaker topology (in contrast to the compactness property).

The above remark that μ-compactness is not a purely topological property is confirmed by the following example, showing that this property is not translated by homeomorphisms. By the above observation (see 2)) the set of Dirac probability measures (single atom measures) on any complete separable metric space endowed with the weak topology is μ-compact and homeomorphic to this space, which is not μ-compact in general.

2.3 Property B

Property B reveals the another relation between the topology and the convex structure of the set $\mathfrak{S}(\mathcal{H})$. The characterization of the analog of this property for arbitrary μ-compact convex set is obtained in theorem 1 in [28].[4] By this theorem property B is equivalent to continuity of the convex hull of any continuous bounded function on the set $\mathfrak{S}(\mathcal{H})$ and to openness of the map $\mathfrak{S}(\mathcal{H})\times\mathfrak{S}(\mathcal{H})\times[0,1]\ni(\rho,\sigma,\lambda)\;\mapsto\;\lambda\rho+(1-\lambda)\sigma\in\mathfrak{S}(\mathcal{H})$. The analog of the last property for arbitrary convex set seems to be simplest for verification and (its equivalent but formally stronger form) is called the *stability* property (see [17], [8] and references therein). In $\mathbb{R}^2$ stability holds for arbitrary convex compact set, in $\mathbb{R}^3$ it is equivalent to closeness of the set of extreme points while in $\mathbb{R}^n, n>3$, it does not follow from the last property [4]. The full characterization of the stability property in finite dimensions is obtained in [17] . In infinite dimensions stability is proved for unit balls in some Banach spaces, in particular, in all strictly convex Banach spaces[5] and for the positive part of the unit ball in all strictly convex Banach lattice [8]. The Banach space $\mathfrak{T}(\mathcal{H})$ of all trace class operators is not strictly convex, but stability property for its subset $\mathfrak{T}_1(\mathcal{H})$ can be easily derived from stability of the set $\mathfrak{S}(\mathcal{H})$ proved in [24] (lemma 3).

[4]This theorem is a partial noncompact generalization of the results in [4], concerning compact convex sets. The full generalization of these results to the class of μ-compact convex sets is obtained in [20].

[5]A Banach space is called strictly convex if its unit ball is strictly convex.

3 The convex hulls and the convex roofs

3.1 Several notions of convexity of a function

In what follows we will consider functions on the set $\mathfrak{S}(\mathcal{H})$ with the range $[-\infty, +\infty]$, which are *semibounded* (lower or upper bounded) on this set.

A semibounded function f on the set $\mathfrak{S}(\mathcal{H})$ is called *convex* if

$$f\left(\sum_i \pi_i\rho_i\right) \leq \sum_i \pi_i f(\rho_i)$$

for arbitrary *finite* ensemble $\{\pi_i, \rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$.

We will use the following two stronger forms of convexity.

A semibounded function f on the set $\mathfrak{S}(\mathcal{H})$ is called σ-*convex* if

$$f\left(\sum_i \pi_i\rho_i\right) \leq \sum_i \pi_i f(\rho_i)$$

for arbitrary *countable* ensemble $\{\pi_i, \rho_i\}$ of states in $\mathfrak{S}(\mathcal{H})$.

A semibounded universally measurable[6] function f on the set $\mathfrak{S}(\mathcal{H})$ is called μ-*convex* if

$$f\left(\int_{\mathfrak{S}(\mathcal{H})} \rho\mu(d\rho)\right) \leq \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu(d\rho)$$

for arbitrary measure μ in $\mathcal{P}$ – *continuous* ensemble of states in $\mathfrak{S}(\mathcal{H})$.

It is clear that

$$\mu\text{-convexity} \;\Rightarrow\; \sigma\text{-convexity} \;\Rightarrow\; \text{convexity}$$

for any universally measurable lower bounded function f.

The simplest example of a convex Borel function on the set $\mathfrak{S}(\mathcal{H})$, which is not σ-convex and μ-convex, is the function taking the value 0 on the convex set of finite rank states and the value $+\infty$ on set of infinite rank states. Difference between the above convexity properties can be also illustrated by the functions $\mathrm{co}H$ (which is convex but not σ-convex) and $\sigma\text{-co}\,\mathrm{Ind}_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})\setminus\mathcal{A}_s}$ (which is σ-convex and bounded but not μ-convex) in examples 1 and 2 below.

[6]This means that the function f is measurable with respect to any measure in $\mathcal{P}$ [21].

By the discrete Yensen's inequality (proposition A-1 in the Appendix) convexity implies σ-convexity for any upper bounded function on the set $\mathfrak{S}(\mathcal{H})$.

By the integral Yensen's inequality (proposition A-2 in the Appendix) all these convexity properties are equivalent for the classes of lower semicontinuous functions and of upper bounded upper semicontinuous functions on the set $\mathfrak{S}(\mathcal{H})$.

3.2 The convex hulls and the convex closure

The *convex hull* of a semibounded function f on the set $\mathfrak{S}(\mathcal{H})$ is defined as the greatest convex function majorized by f [14, 22], which means that

$$\mathrm{co}f(\rho) = \inf_{\{\pi_i,\rho_i\}\in\mathcal{P}^{\mathrm{f}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \rho\in\mathfrak{S}(\mathcal{H}), \tag{1}$$

(where the infimum is over all finite ensembles $\{\pi_i,\rho_i\}$ of states with the average state ρ).

The σ*-convex hull* of a semibounded function f on the set $\mathfrak{S}(\mathcal{H})$ is defined as follows

$$\sigma\text{-}\mathrm{co}f(\rho) = \inf_{\{\pi_i,\rho_i\}\in\mathcal{P}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \rho\in\mathfrak{S}(\mathcal{H}), \tag{2}$$

(where the infimum is over all countable ensembles $\{\pi_i,\rho_i\}$ of states with the average state ρ). The function σ-co$f$ is σ-convex since for any countable ensemble $\{\lambda_i,\sigma_i\}$ with the average state σ and any family $\{\{\pi_{ij},\rho_{ij}\}_j\}_i$ of countable ensembles such that $\sigma_i=\sum_j \pi_{ij}\rho_{ij}$ for all i the countable ensemble $\{\lambda_i\pi_{ij},\rho_{ij}\}_{ij}$ has the average state σ. Thus σ-co$f$ is the greatest σ-convex function majorized by f.

The μ*-convex hull* of a Borel semibounded function f on the set $\mathfrak{S}(\mathcal{H})$ is defined as follows

$$\mu\text{-}\mathrm{co}f(\rho) = \inf_{\mu\in\mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma), \quad \rho\in\mathfrak{S}(\mathcal{H}), \tag{3}$$

(where the infimum is over all probability measures μ with the barycenter ρ). If the function μ-co$f$ is universally measurable[7] and μ-convex then it is the greatest μ-convex function majorized by f. By propositions 1 and 2

[7] By using the results in [21] this can be proved for any bounded Borel function f.

below (taking with evident convexity of the function μ-co$f$ and proposition A-2 in the Appendix) this holds if the function f is lower bounded lower semicontinuous or upper bounded upper semicontinuous.

The *convex closure* $\overline{\mathrm{co}}f$ of a lower bounded function f on the set $\mathfrak{S}(\mathcal{H})$ is defined as the greatest convex lower semicontinuous (closed) function majorized by f [14]. By the Fenchel theorem (see [14],[22]) the function $\overline{\mathrm{co}}f$ coincides with the double Fenchel transformation of the function f, which means that[8]

$$\overline{\mathrm{co}}f(\rho) = f^{**}(\rho) = \sup_{A\in\mathfrak{B}_+(\mathcal{H})} \left[\mathrm{Tr}A\rho - f^*(A)\right], \quad \rho\in\mathfrak{S}(\mathcal{H}), \tag{4}$$

where

$$f^*(A) = \sup_{\rho\in\mathfrak{S}(\mathcal{H})} \left[\mathrm{Tr}A\rho - f(\rho)\right], \quad A\in\mathfrak{B}_+(\mathcal{H}).$$

It follows from the definitions and proposition A-2 in the Appendix that

$$\overline{\mathrm{co}}f(\rho) \leq \mu\text{-co}f(\rho) \leq \sigma\text{-co}f(\rho) \leq \mathrm{co}f(\rho), \quad \rho\in\mathfrak{S}(\mathcal{H}),$$

for arbitrary Borel lower bounded function f on the set $\mathfrak{S}(\mathcal{H})$. It is possible to prove (see corollary 1 below) that the equalities hold in the above inequality for arbitrary continuous bounded function f on the set $\mathfrak{S}(\mathcal{H})$. The following examples show that the last assertion is not true in general.

Example 1. Let H be the von Neumann entropy and ρ_0 be a state such that $H(\rho_0) = +\infty$. Since the set of states with finite entropy is convex co$H(\rho_0) = +\infty$ while the spectral theorem implies σ-co$H(\rho_0) = 0$.

Example 2. Let f be the indicator function of the complement of the closed set $\mathcal{A}_s$ of pure product states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$ and ω_0 be the separable state constructed in [12] such that any measure in $\mathcal{P}_{\{\omega_0\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{H}))$ have no atoms in $\mathcal{A}_s$. It is easy to show that σ-co$f(\omega_0) = 1$, but μ-co$f(\omega_0) = 0$ since lemma 1 in [12] implies existence of a measure μ_0 in $\mathcal{P}_{\{\omega_0\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{H}))$ supported by the set $\mathcal{A}_s$. Note that σ-co$f$ is a μ_0-integrable σ-convex bounded function on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$, for which Jensen's inequality does not hold:

$$1 = \sigma\text{-co}f(\omega_0) > \int_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})} \sigma\text{-co}f(\omega)\mu_0(d\omega) = 0$$

[8] To obtain the below expression from the Fenchel theorem it is necessary to consider the extension $\hat{f}$ of the function f to the real Banach space $\mathfrak{T}_h(\mathcal{H})$ by setting $\hat{f} = +\infty$ on $\mathfrak{T}_h(\mathcal{H})\setminus\mathfrak{S}(\mathcal{H})$ and to use coincidence of the space $\mathfrak{B}_h(\mathcal{H})$ with the dual space of $\mathfrak{T}_h(\mathcal{H})$.

(since the functions σ-co$f$ and f coincide on the support of the measure μ_0).

Example 3. Let f be the indicator function of a set consisting of one pure state. Then μ-co$f = f$ while $\overline{\text{co}}f \equiv 0$.

Property A of the set $\mathfrak{S}(\mathcal{H})$ implies the following result ([27], theorem 1).

Proposition 1. *Let f be a lower semicontinuous lower bounded function f on the set $\mathfrak{S}(\mathcal{H})$. Then the μ-convex hull of this function is lower semicontinuous, which means that*

$$\overline{\text{co}}f(\rho) = \mu\text{-co}f(\rho) = \inf_{\mu \in \mathcal{P}_{\{\rho\}}} \int_{\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma), \quad \rho \in \mathfrak{S}(\mathcal{H}), \tag{5}$$

where the infimum is achieved at some measure in $\mathcal{P}_{\{\rho\}}$.

Property A is an essential condition of validity of representation (5) for the convex closure. This is confirmed by the following observation.

Remark 1. *The analog of representation (5) holds for any lower semicontinuous lower bounded function f on arbitrary bounded convex closed subset of the positive cone of the Shatten class of order p if and only if $p = 1$. Moreover, if $p > 1$ then the analog of representation (5) is not valid in general even for concave continuous bounded function f.*

The assertion concerning the case $p = 1$ follows from proposition 3 in [28]. As an example for the case $p > 1$ one can consider the function $f(\cdot) = 1 - \|\cdot\|_p$ on the positive part $\mathcal{A}_p$ of the unit ball of the Shatten class of order $p > 1$. Since the extreme point 0 of the set $\mathcal{A}_p$ can be approximated in the $\|\cdot\|_p$-norm topology by convex combinations of operators in $\mathcal{A}_p$ with the unit norm we have $\overline{\text{co}}f(0) = 0$ while μ-co$f(0) = f(0) = 1$. $\square$

Representation (5) implies, in particular, that the convex closure of arbitrary lower semicontinuous lower bounded function on the set $\mathfrak{S}(\mathcal{H})$ coincides with this function on the set extr$\mathfrak{S}(\mathcal{H})$ of pure states. The simple modification of the example in remark 1 shows that this coincidence does not hold in general even for concave continuous bounded function on noncompact simplex with closed countable set of isolated extreme points.

Note also that the condition of lower boundedness in proposition 1 can not be dropped since by lemma 2 below any convex lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ is either lower bounded or does not take finite values.

Property B of the set $\mathfrak{S}(\mathcal{H})$ implies the following result.

Proposition 2. *Let f be an upper semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$. Then the convex hull* co$f$ *of this function is upper semicontinuous.*

If in addition the function f is upper bounded then the convex hull, the σ-convex hull and the μ-convex hull of this function coincide:

$$\mathrm{co}f = \sigma\text{-}\mathrm{co}f = \mu\text{-}\mathrm{co}f.$$

Proof. Upper semicontinuity of the function $\mathrm{co}f$ can be proved by using the more general assertion of lemma 4 in section 4 since for arbitrary sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$, converging to a state ρ_0, lemma 3 in [11] implies existence of such $\mathfrak{H}$-operator H in the space $\mathcal{H}$ that $\sup_{n\geq 0}\mathrm{Tr}H\rho_n < +\infty$.

Coincidence of the functions $\mathrm{co}f$ and μ-$\mathrm{co}f$ under the condition of upper boundedness of the function f is easily proved by using upper semicontinuiuty of the function $\mu \mapsto \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu(d\rho)$ on the set $\mathcal{P}(\mathfrak{S}(\mathcal{H}))$ and density of measures with finite support in the set of all measures with given barycenter (lemma 1 in [11]). $\square$

The above example 3 shows that the condition of proposition 2 does not imply coincidence of the function $\overline{\mathrm{co}}f$ with the function μ-$\mathrm{co}f = \sigma$-$\mathrm{co}f = \mathrm{co}f$.

The above two propositions have the following obvious corollary.

Corollary 1. *Let f be a continuous lower bounded function on the set $\mathfrak{S}(\mathcal{H})$. Then the convex hull $\mathrm{co}f$ is continuous on any subset of $\mathfrak{S}(\mathcal{H})$, where it coincides with the μ-convex hull μ-$\mathrm{co}f$.*

If in addition the function f is upper bounded then

- *the convex hull, the σ-convex hull, the μ-convex hull and the convex closure of the function f coincide: $\overline{\mathrm{co}}f = \mu$-$\mathrm{co}f = \sigma$-$\mathrm{co}f = \mathrm{co}f$;*

- *the function $\overline{\mathrm{co}}f = \mu$-$\mathrm{co}f = \sigma$-$\mathrm{co}f = \mathrm{co}f$ is continuous.*

By using proposition 1 it is easy to show that (under the condition of the first part of corollary 1) the necessary and sufficient condition of coincidence of the functions $\mathrm{co}f$ and μ-$\mathrm{co}f$ at a state $\rho_0 \in \mathfrak{S}(\mathcal{H})$ consists in validity of the Jensen's inequality $\mathrm{co}f(\rho_0) \leq \int \mathrm{co}f(\rho)\mu(d\rho)$ for any measure μ in $\mathcal{P}_{\{\rho_0\}}$ (the convex function $\mathrm{co}f$ is Borel by proposition 2). The particular sufficient condition of this coincidence is considered in corollary 6 in section 4.

We will use the following approximation result.

Lemma 1. *Let f be a Borel lower bounded function on the set $\mathfrak{S}(\mathcal{H})$. For arbitrary state ρ_0 in $\mathfrak{S}(\mathcal{H})$ there exists a sequence $\{\rho_n\}$, converging to the state ρ_0, such that*

$$\limsup_{n\to+\infty} \sigma\text{-}\mathrm{co}f(\rho_n) \leq \limsup_{n\to+\infty} \mathrm{co}f(\rho_n) \leq \mu\text{-}\mathrm{co}f(\rho_0).$$

If in addition the function f is lower semicontinuous then

$$\lim_{n\to+\infty}\sigma\text{-co}f(\rho_n)=\lim_{n\to+\infty}\mathrm{co}f(\rho_n)=\mu\text{-co}f(\rho_0).$$

Proof. It is sufficient to consider the case of nonnegative function f. For given natural n let μ_n be a measure in $\mathcal{P}_{\{\rho_0\}}$ such that

$$\mu\text{-co}f(\rho_0)\geq\int_{\mathfrak{S}(\mathcal{H})}f(\rho)\mu_n(d\rho)-1/n.$$

Since the set $\mathfrak{S}(\mathcal{H})$ is separable there exists sequence $\{\mathcal{A}_i^n\}$ of Borel subsets of $\mathfrak{S}(\mathcal{H})$ with diameter $\leq 1/n$ such that $\mathfrak{S}(\mathcal{H})=\bigcup_i\mathcal{A}_i^n$ and $\mathcal{A}_i^n\cap\mathcal{A}_j^n=\emptyset$ if $j\neq i$. Let $m=m(n)$ be such number that $\sum_{i=m+1}^{+\infty}\mu_n(\mathcal{A}_i^n)<1/n$. Without loss of generality we may assume that $\mu_n(\mathcal{A}_i^n)>0$ for $i=\overline{1,m}$. For each i the set $\mathcal{A}_i^n$ contains a state ρ_i^n such that $f(\rho_i^n)\leq(\mu_n(\mathcal{A}_i^n))^{-1}\int_{\mathcal{A}_i^n}f(\rho)\mu_n(d\rho)$.

Let $\mathcal{B}_n=\bigcup_{i=1}^m\mathcal{A}_i^n$. Consider the state $\rho_n=(\mu_n(\mathcal{B}_n))^{-1}\sum_{i=1}^m\mu_n(\mathcal{A}_i^n)\rho_i^n$. We want to show that

$$\lim_{n\to+\infty}\rho_n=\rho_0. \tag{6}$$

For each i the state $\hat{\rho}_i^n=(\mu_n(\mathcal{A}_i^n))^{-1}\int_{\mathcal{A}_i^n}\rho\mu_n(d\rho)$ lies in the set $\overline{\mathrm{co}}(\mathcal{A}_i^n)$ with diameter $\leq 1/n$. It follows that $\|\rho_i^n-\hat{\rho}_i^n\|_1\leq 1/n$ for $i=\overline{1,m}$. By noting that $\mu_n(\mathcal{B}_n)=\sum_{i=1}^m\mu_n(\mathcal{A}_i^n)$ we have

$$\begin{aligned}
&\|\rho_n-\rho_0\|_1=\\
&\|(\mu_n(\mathcal{B}_n))^{-1}\sum_{i=1}^m\mu_n(\mathcal{A}_i^n)\rho_i^n-\sum_{i=1}^m\int_{\mathcal{A}_i^n}\rho\mu_n(d\rho)-\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{B}_n}\rho\mu_n(d\rho)\|_1\\
&\leq\sum_{i=1}^m\mu_n(\mathcal{A}_i^n)\|(\mu_n(\mathcal{B}_n))^{-1}\rho_i^n-\hat{\rho}_i^n\|_1+\|\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{B}_n}\rho\mu_n(d\rho)\|_1\\
&\leq(1-\mu_n(\mathcal{B}_n))+\sum_{i=1}^m\mu_n(\mathcal{A}_i^n)\|\rho_i^n-\hat{\rho}_i^n\|_1+\mu_n(\mathfrak{S}(\mathcal{H})\setminus\mathcal{B}_n)<3/n,
\end{aligned}$$

which implies (6).

By the choice of the states ρ_i^n we have

$$\begin{aligned} \mathrm{co}f(\rho_n) &\leq (\mu_n(\mathcal{B}_n))^{-1}\sum_{i=1}^{m}\mu_n(\mathcal{A}_i^n)f(\rho_i^n) \\ &\leq (\mu_n(\mathcal{B}_n))^{-1}\sum_{i=1}^{m}\int_{\mathcal{A}_i^n} f(\rho)\mu_n(d\rho) \\ \leq (\mu_n(\mathcal{B}_n))^{-1}&\int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_n(d\rho) \leq (1-1/n)^{-1}(\mu\text{-co}f(\rho_0)+1/n). \end{aligned}$$

This implies the first assertion of the lemma. By proposition 1 the second assertion follows from the first one (since σ-co$f \geq \mu$-co$f = \overline{\mathrm{co}}f$). $\square$

We will also use the following corollary of boundedness of the set $\mathfrak{S}(\mathcal{H})$ as a subset of $\mathfrak{T}(\mathcal{H})$.

Lemma 2. *Let f be a concave upper semicontinuous function on convex subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$. If the function f is finite at a particular state in $\mathcal{A}$ then it is upper bounded on the set $\mathcal{A}$.*

Proof. Let ρ_0 be such state in $\mathcal{A}$ that $f(\rho_0) = c_0 \neq \pm\infty$. Without loss of generality we can consider that $c_0 = 0$. If there exists a sequence $\{\rho_n\} \subset \mathcal{A}$ such that $\lim_{n\to+\infty} f(\rho_n) = +\infty$ then the sequence of states $\sigma_n = (1-\lambda_n)\rho_0 + \lambda_n\rho_n$ in $\mathcal{A}$, where $\lambda_n = (f(\rho_n))^{-1}$, converges to the state ρ_0 by boundedness of the set $\mathcal{A}$ and $f(\sigma_n) \geq \lambda_n f(\rho_n) = 1$ by concavity of the function f, contradicting to upper semicontinuity of this function. $\square$

3.3 The convex roofs

In the case $\dim\mathcal{H} < +\infty$ any state in $\mathfrak{S}(\mathcal{H})$ can be represented as the average state of some finite ensemble of pure states. This provides correctness of the following convex extension to the set $\mathfrak{S}(\mathcal{H})$ of an arbitrary function f defined on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$ of pure states

$$f_*(\rho) = \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}^{\mathrm{f}}_{\{\rho\}}}\sum_i \pi_i f(\rho_i), \quad \rho \in \mathfrak{S}(\mathcal{H}), \tag{7}$$

(where the infimum is over all finite ensembles $\{\pi_i, \rho_i\}$ of *pure* states with the average state ρ). Following [30] we will call this extension the *convex roof* of the function f. The notion of convex roof plays essential role in quantum information theory, where it is used in particular for construction of entanglement monotones (see section 5 below).

In the case $\dim\mathcal{H} = +\infty$ we can consider the following two generalizations of the above construction.

The *σ-convex roof* of a semibounded function f on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$ of pure states is the function f_*^σ on the set $\mathfrak{S}(\mathcal{H})$ defined as follows

$$f_*^\sigma(\rho) = \inf_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}} \sum_i \pi_i f(\rho_i), \quad \rho\in\mathfrak{S}(\mathcal{H}), \tag{8}$$

(where the infimum is over all countable ensembles $\{\pi_i, \rho_i\}$ of *pure* states with the average state ρ). Similar to the case of function $\sigma\text{-co}f$ it is easy to show σ-convexity of the function f_*^σ. Thus f_*^σ is the greatest σ-convex extension of the function f to the set $\mathfrak{S}(\mathcal{H})$.

The *μ-convex roof* of a semibounded Borel function f on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$ of pure states is the function f_*^μ on the set $\mathfrak{S}(\mathcal{H})$ defined as follows

$$f_*^\mu(\rho) = \inf_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}} \int_{\mathrm{extr}\mathfrak{S}(\mathcal{H})} f(\sigma)\mu(d\sigma), \quad \rho\in\mathfrak{S}(\mathcal{H}), \tag{9}$$

(where the infimum is over all probability measures μ supported by *pure* states with the barycenter ρ). If the function f_*^μ is universally measurable[9] and μ-convex then it is the greatest μ-convex extension of the function f to the set $\mathfrak{S}(\mathcal{H})$. By propositions 3 and 4 below (taking with evident convexity of the function f_*^μ and proposition A-2 in the Appendix) this holds if the function f is lower bounded lower semicontinuous or upper bounded upper semicontinuous.

Property A of the set $\mathfrak{S}(\mathcal{H})$ (in fact, of the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$) implies the following result ([27], theorem 2).

Proposition 3. *Let f be an arbitrary lower semicontinuous lower bounded function on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$. Then*

- *the function f_*^μ is the greatest lower semicontinuous convex extension of the function f to the set $\mathfrak{S}(\mathcal{H})$;*
- *for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ the infimum in the definition of the value $f_*^\mu(\rho)$ in (9) is achieved at some measure in $\widehat{\mathcal{P}}_{\{\rho\}}(\mathfrak{S}(\mathcal{H}))$.*

[9] By using the results in [21] this can be proved for any bounded Borel function f.

Importance of property A in the proof of this proposition is illustrated by the following observation.

Remark 2. *The assertion of proposition 3 holds for any bounded convex closed subset $\mathcal{A}$ (in the role of the set $\mathfrak{S}(\mathcal{H})$) of the positive cone of the Shatten class of order p with closed*[10] *set* $\mathrm{extr}\mathcal{A}$ *if and only if* $p = 1$. *Moreover, if* $p > 1$ *then this assertion is not valid in general even for continuous bounded function* f.

The assertion concerning the case $p = 1$ follows from theorem 2 in [28]. In the case $p > 1$ the obvious modification of the example in remark 1 shows existence of continuous bounded function on the closed set $\mathrm{extr}\,\mathcal{A}_p$ having no convex lower semicontinuous extensions to the set $\mathcal{A}_p = \overline{\mathrm{co}}(\mathrm{extr}\mathcal{A}_p)$. $\square$

Property B of the set $\mathfrak{S}(\mathcal{H})$ implies the following result ([27], theorem 2).

Proposition 4. *Let f be an upper semicontinuous upper bounded function on the set* $\mathrm{extr}\mathfrak{S}(\mathcal{H})$. *Then*

- *the σ-convex roof and the μ-convex roof of the function f coincide:*

$$f_*^{\sigma} = f_*^{\mu};$$

- *the function $f_*^{\sigma} = f_*^{\mu}$ is upper semicontinuous on the set $\mathfrak{S}(\mathcal{H})$ and coincides with the greatest upper bounded convex extension of the function f to this set.*

The above two propositions have the following obvious corollary.

Corollary 2. *Let f be a continuous bounded function on the set* $\mathrm{extr}\mathfrak{S}(\mathcal{H})$. *Then the function $f_*^{\sigma} = f_*^{\mu}$ is continuous on the set $\mathfrak{S}(\mathcal{H})$.*

By this corollary for arbitrary continuous bounded function on the set of pure states there exists of at least one continuous bounded convex extension to the set of all states.[11]

[10] By using this condition and analog of property A for the set $\mathcal{A}$ one can prove (see [20]) that any element of the set $\mathcal{A}$ can be represented as the barycenter of some probability measure supported by the set $\mathrm{extr}\mathcal{A}$, which implies correctness of the definition of the μ-convex roof.

[11] The analogous assertion for a compact convex set (in the role of $\mathfrak{S}(\mathcal{H})$) is equivalent to closeness of the set of its extreme points [13, corollary 2], but this result is not valid for general noncompact subsets of a separable Banach space.

3.4 The convex hulls of concave functions

In the case $\dim\mathcal{H} < +\infty$ it is easy to show that the convex hull of arbitrary concave function f defined on the set $\mathfrak{S}(\mathcal{H})$ coincides with the convex roof of the restriction $f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}$ of this function to the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$. By stability of the set $\mathfrak{S}(\mathcal{H})$ (property B) continuity of the function f implies continuity of the function $\mathrm{co}f = (f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})_*$.

In the case $\dim\mathcal{H} = +\infty$ the analog of this observations is established in the following proposition.

Proposition 5. *Let f be a concave semibounded function on the set $\mathfrak{S}(\mathcal{H})$*

If the function f is lower bounded then σ-co$f = \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\sigma$. If in addition the function f is lower semicontinuous then μ-co$f = \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\mu$ and this function is lower semicontinuous.

If the function f is upper semicontinuous (correspondingly continuous and lower bounded) then

$$\mathrm{co}f = \sigma\text{-}\mathrm{co}f = \mu\text{-}\mathrm{co}f = (f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})_*^\sigma = (f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})_*^\mu$$

and this function is upper semicontinuous (correspondingly continuous).

Proof. To show coincidence of the functions σ-co$f$ and $\left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\sigma$ (correspondingly of the functions μ-co$f$ and $\left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\mu$) it is sufficient to prove the inequality σ-co$f \geq \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\sigma$ (correspondingly the inequality μ-co$f \geq \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\mu$).

The fist inequality for the concave lower bounded function f directly follows from the discrete Yensen's inequality (proposition A-1 in the Appendix).

Let f be a lower bounded lower semicontinuous concave function and ρ_0 be an arbitrary state. By lemma 1 there exists sequence $\{\rho_n\}$, converging to the state ρ_0, such that $\lim_{n\to+\infty}\sigma$-co$f(\rho_n) = \mu$-co$f(\rho_0)$. By the above assertion we have

$$\sigma\text{-}\mathrm{co}f(\rho_n) = \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\sigma(\rho_n) \geq \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\mu(\rho_n), \quad \forall n.$$

By proposition 3 the function $\left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\mu$ is lower semicontinuous. Hence passing to the limit $n \to +\infty$ in the above inequality leads to the inequality μ-co$f(\rho_0) \geq \left(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}\right)_*^\mu(\rho_0)$. This implies the first assertion of the proposition.

Let f be an upper semicontinuous concave function taking not only the values $\pm\infty$. By lemma 2 this function is upper bounded. Propositions 2 and

4 imply respectively $\mathrm{co}f = \sigma\text{-}\mathrm{co}f = \mu\text{-}\mathrm{co}f$ and $(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})^{\sigma}_{*} = (f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})^{\mu}_{*}$ as well as upper semicontinuity of these functions. Since $\mathrm{co}f \geq (f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}$ by proposition A-2 in the Appendix and $\mu\text{-}\mathrm{co}f \leq (f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})^{\mu}_{*}$ by the definitions, this implies the second assertion of the proposition.

The assertion concerning concave continuous lower bounded function f follows from the previous ones. □

3.5 One result concerning the convex closure

It is well known that for arbitrary increasing sequence $\{f_n\}$ of continuous functions on a convex compact set $\mathcal{A}$, pointwise converging to a continuous function f_0, the corresponding sequence $\{\overline{\mathrm{co}}f_n\}$ converges to the function $\overline{\mathrm{co}}f_0$.[12] It turns out that μ-compactness of the set $\mathfrak{S}(\mathcal{H})$ (property A) implies (in fact, *means*, see remark 3 below) the analogous observation.

Proposition 6. *For arbitrary increasing sequence $\{f_n\}$ of lower semicontinuous lower bounded functions on the set $\mathfrak{S}(\mathcal{H})$ and arbitrary converging sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$ the following inequality holds*

$$\liminf_{n\to+\infty} \overline{\mathrm{co}}f_n(\rho_n) \geq \overline{\mathrm{co}}f_0(\rho_0), \quad \textit{where} \quad f_0 = \lim_{n\to+\infty} f_n \quad \textit{and} \quad \rho_0 = \lim_{n\to+\infty} \rho_n.$$

In particular

$$\lim_{n\to+\infty} \overline{\mathrm{co}}f_n(\rho) = \overline{\mathrm{co}}f_0(\rho), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}).$$

Remark 3. Property A of the set $\mathfrak{S}(\mathcal{H})$ can be derived from validity of the last assertion of proposition 6. Moreover, the following stronger version of this statement can be proved (see Appendix 7.2).

Let $\mathcal{A}$ be a convex bounded closed subset of a separable Banach space. If for arbitrary increasing sequence $\{f_n\}$ of concave continuous bounded functions on the set $\mathcal{A}$ with continuous bounded pointwise limit f_0 the sequence $\{\overline{\mathrm{co}}f_n\}$ pointwise converges to the function $\overline{\mathrm{co}}f_0$ then the analog of property A holds for the set $\mathcal{A}$. □

Proof of proposition 6. For arbitrary Borel function g on the set $\mathfrak{S}(\mathcal{H})$ and arbitrary measure $\mu \in \mathcal{P}$ we will use the following notation:

$$\mu(g) = \int_{\mathfrak{S}(\mathcal{H})} g(\sigma)\mu(d\sigma).$$

[12] It follows from Dini's lemma. The importance of the compactness condition can be shown by the sequence of the functions $f_n(x) = \exp(-x^2/n)$ on $\mathbb{R}$, converging to the function $f_0(x) \equiv 1$, such that $\overline{\mathrm{co}}f_n(x) \equiv 0$ for all n.

Without loss of generality we may assume that the sequence $\{f_n\}$ consists of nonnegative functions. Suppose there exists such sequence $\{\rho_n\}$, converging to the state ρ_0, that[13]

$$\overline{\mathrm{co}} f_n(\rho_n) \leq \overline{\mathrm{co}} f_0(\rho_0) - \Delta, \quad \Delta > 0, \quad \forall n.$$

By representation (4) there exists such continuous affine function α on the set $\mathfrak{S}(\mathcal{H})$ that

$$\alpha(\rho) \leq f_0(\rho), \quad \forall \rho \in \mathfrak{S}(\mathcal{H}), \quad \text{and} \quad \overline{\mathrm{co}} f_0(\rho_0) \leq \alpha(\rho_0) + \tfrac{1}{4}\Delta. \tag{10}$$

Let N be such number that $|\alpha(\rho_n) - \alpha(\rho_0)| < \frac{1}{4}\Delta$ for all $n \geq N$.

By proposition 1 for each n there exists such measure $\mu_n \in \mathcal{P}_{\{\rho_n\}}$ that $\overline{\mathrm{co}} f_n(\rho_n) = \mu_n(f_n)$. Since the function α is affine we have

$$\mu_n(\alpha) - \mu_n(f_n) = \alpha(\rho_n) - \overline{\mathrm{co}} f_n(\rho_n)$$

$$= [\,\alpha(\rho_n) - \alpha(\rho_0)\,] + [\,\alpha(\rho_0) - \overline{\mathrm{co}} f_0(\rho_0)\,] + [\,\overline{\mathrm{co}} f_0(\rho_0) - \overline{\mathrm{co}} f_n(\rho_n)\,] \tag{11}$$

$$\geq -\tfrac{1}{4}\Delta - \tfrac{1}{4}\Delta + \Delta = \tfrac{1}{2}\Delta, \quad \forall n \geq N.$$

Property A of the set $\mathfrak{S}(\mathcal{H})$ implies relative compactness of the sequence $\{\mu_n\}$. Hence by Prokhorov theorem (see [3],[18]) this sequence is *tight*, which means existence of such compact subset $\mathcal{K}_\varepsilon \subset \mathfrak{S}(\mathcal{H})$ for each $\varepsilon > 0$ that $\mu_n(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{K}_\varepsilon) < \varepsilon$ for all n.

Let $M = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} |\alpha(\rho)|$ and $\varepsilon_0 = \frac{\Delta}{4M}$. By (11) for all $n \geq N$ we have

$$\int_{\mathcal{K}_{\varepsilon_0}} (\alpha(\rho) - f_n(\rho))\mu_n(d\rho) \geq \tfrac{1}{2}\Delta - \int_{\mathfrak{S}(\mathcal{H}) \setminus \mathcal{K}_{\varepsilon_0}} (\alpha(\rho) - f_n(\rho))\mu_n(d\rho) \geq \tfrac{1}{4}\Delta.$$

Hence, the set $\mathcal{C}_n = \{\rho \in \mathcal{K}_{\varepsilon_0} \,|\, \alpha(\rho) \geq f_n(\rho) + \frac{1}{4}\Delta\}$ is nonempty for all $n \geq N$.

Since the sequence $\{f_n\}$ is increasing the sequence $\{\mathcal{C}_n\}$ of *closed* subsets of the *compact* set $\mathcal{K}_{\varepsilon_0}$ is monotone: $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$, $\forall n$. Hence there exists $\rho_* \in \bigcap_n \mathcal{C}_n$. This means that $\alpha(\rho_*) \geq f_n(\rho_*) + \frac{1}{4}\Delta$ for all n, and hence $\alpha(\rho_*) > f_0(\rho_*)$, contradicting to (10). $\square$

Corollary 3. *For arbitrary increasing sequence $\{f_n\}$ of lower semicontinuous lower bounded functions on the set* $\mathrm{extr}\mathfrak{S}(\mathcal{H})$ *and arbitrary converging sequence $\{\rho_n\}$ of states in $\mathfrak{S}(\mathcal{H})$ the following inequality holds*

$$\liminf_{n\to+\infty} (f_n)_*^\mu(\rho_n) \geq (f_0)_*^\mu(\rho_0), \quad \textit{where} \quad f_0 = \lim_{n\to+\infty} f_n \quad \textit{and} \quad \rho_0 = \lim_{n\to+\infty} \rho_n.$$

[13] We assume below that $f_0(\rho_0) < +\infty$. The case $f_0(\rho_0) = +\infty$ is considered similarly.

In particular
$$\lim_{n\to+\infty}(f_n)^{\mu}_{*}(\rho)=(f_0)^{\mu}_{*}(\rho),\quad \forall\rho\in\mathfrak{S}(\mathcal{H}).$$

Proof. By theorem 2 in [27] for every lower semicontinuous lower bounded function f on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$ the function

$$f^{*}(\rho)=\sup_{\mu\in\widehat{\mathcal{P}}_{\{\rho\}}}\int_{\mathrm{extr}\mathfrak{S}(\mathcal{H})}f(\sigma)\mu(d\sigma)=\sup_{\{\pi_i,\rho_i\}\in\widehat{\mathcal{P}}_{\{\rho\}}}\sum_i\pi_i f(\rho_i),\quad \rho\in\mathfrak{S}(\mathcal{H}),$$

is a lower semicontinuous lower bounded concave extension of the function f to the set $\mathfrak{S}(\mathcal{H})$. It is clear that for arbitrary increasing sequence $\{f_n\}$ of lower semicontinuous lower bounded functions on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$, converging to the function f_0, the corresponding increasing sequence $\{f^{*}_n\}$ converges to the function f^{*}_0 on the set $\mathfrak{S}(\mathcal{H})$. Thus the assertion of the corollary can be derived from proposition 6 by using propositions 1 and 5.$\square$

Remark 4. The μ-convex roof can not be changed by the σ-convex roof in corollary 3. Indeed, let f be the characteristic function of the set $\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})\setminus\mathcal{A}_s$ and ω_0 be the separable state considered in example 2. By the proof of lemma 1 in [27] this function f can be represented as a limit of the increasing sequence $\{f_n\}$ of continuous bounded functions on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{H})$. Since by corollary 2 $(f_n)^{\sigma}_{*}=(f_n)^{\mu}_{*}$ for all n, corollary 3 and the property of the state ω_0 imply

$$\lim_{n\to+\infty}(f_n)^{\sigma}_{*}(\omega_0)=(f_0)^{\mu}_{*}(\omega_0)=0,\quad \text{while}\quad (f_0)^{\sigma}_{*}(\omega_0)=1.$$

Remark 5. The monotonous convergence theorem implies the following results dual to the second assertions of proposition 6 and of corollary 3.

For arbitrary decreasing sequence $\{f_n\}$ of upper semicontinuous uniformly upper bounded functions on the set $\mathfrak{S}(\mathcal{H})$ the following relation holds

$$\lim_{n\to+\infty}\mu\text{-co}f_n(\rho)=\mu\text{-co}f_0(\rho),\quad \forall\rho\in\mathfrak{S}(\mathcal{H}),\quad \textit{where}\quad f_0=\lim_{n\to+\infty}f_n.$$

For arbitrary decreasing sequence $\{f_n\}$ of upper semicontinuous uniformly upper bounded functions on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$ the following relation holds

$$\lim_{n\to+\infty}(f_n)^{\mu}_{*}(\rho)=(f_0)^{\mu}_{*}(\rho),\quad \forall\rho\in\mathfrak{S}(\mathcal{H}),\quad \textit{where}\quad f_0=\lim_{n\to+\infty}f_n.$$

By using corollary 1, proposition 6, the first assertion of remark 5 and Dini's lemma the following result can be easily proved.

Corollary 4. *Let $\{f_t\}_{t\in\mathrm{T}\subseteq\mathbb{R}}$ be a family of continuous bounded functions on the set $\mathfrak{S}(\mathcal{H})$ such that*

- $f_{t_1}(\rho)\leq f_{t_2}(\rho)$ *for all* $\rho\in\mathfrak{S}(\mathcal{H})$ *and all* $t_1,t_2\in\mathrm{T}$ *such that* $t_1<t_2$*;*
- *the function* $\mathrm{T}\ni t\mapsto f_t(\rho)$ *is continuous for all* $\rho\in\mathfrak{S}(\mathcal{H})$*.*

Then the function $\mathfrak{S}(\mathcal{H})\times\mathrm{T}\ni(\rho,t)\mapsto \mathrm{co}f_t(\rho)$ *is continuous.*

By using corollary 2, corollary 6, the second assertion of remark 5 and Dini's lemma the analogous result can be proved for the μ-convex roof of a family of continuous bounded functions on the set $\mathrm{extr}\mathfrak{S}(\mathcal{H})$.

4 The main theorem

Let α be a lower semicontinuous affine function on the set $\mathfrak{S}(\mathcal{H})$ with the range $[0,+\infty]$. Consider the family of closed subsets

$$\mathcal{A}_c=\{\rho\in\mathfrak{S}(\mathcal{H})\,|\,\alpha(\rho)\leq c\},\quad c\in\mathbb{R}_+. \tag{12}$$

In the following theorem the properties of restrictions of convex hulls to the subsets of this family are considered.

Theorem 1. *Let f be a Borel lower bounded function on the set $\mathfrak{S}(\mathcal{H})$ and α be the above affine function. If the function f has upper semicontinuous bounded restriction to the set $\mathcal{A}_c$ for each $c\geq 0$ and*

$$\limsup_{c\to+\infty}\, c^{-1}\sup_{\rho\in\mathcal{A}_c} f(\rho)<+\infty \tag{13}$$

then

$$\mathrm{co}f(\rho)=\sigma\text{-}\mathrm{co}f(\rho)=\mu\text{-}\mathrm{co}f(\rho)$$

for all $\rho\in\bigcup_{c>0}\mathcal{A}_c$ and the common restriction of these functions to the set $\mathcal{A}_c$ is upper semicontinuous for each $c\geq 0$.

If in addition the function f is lower semicontinuous on the set $\mathfrak{S}(\mathcal{H})$ then

$$\mathrm{co}f(\rho)=\sigma\text{-}\mathrm{co}f(\rho)=\mu\text{-}\mathrm{co}f(\rho)=\overline{\mathrm{co}}f(\rho)$$

for all $\rho \in \bigcup_{c>0} \mathcal{A}_c$ *and the common restriction of these functions to the set* $\mathcal{A}_c$ *is continuous for each* $c \geq 0$.

Proof. Without loss of generality we can assume that f is a nonnegative function.

Let ρ_0 be a state such that $\alpha(\rho_0) = c_0 < +\infty$. By the condition $\mu\text{-co}f(\rho_0) \leq f(\rho_0) < +\infty$. Let $\varepsilon > 0$ be arbitrary and μ_0 be such measure in $\mathcal{P}_{\{\rho_0\}}$ that

$$\int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_0(d\rho) < \mu\text{-co}f(\rho_0) + \varepsilon.$$

Condition (13) implies existence of such positive numbers c_* and M that $f(\rho) \leq M\alpha(\rho)$ for all $\rho \in \mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_{c_*}$.

Note that $\lim_{c\to+\infty} \mu_0(\mathcal{A}_c) = 1$. Indeed, it follows from the inequality

$$c\mu_0(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c) \leq \int_{\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho) + \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho) = \alpha(\rho_0) = c_0$$

obtained by using corollary A in the Appendix that

$$\mu_0(\mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c) \leq \frac{c_0}{c}.$$

Thus the monotonous convergence theorem implies

$$\lim_{c\to+\infty} \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho) = \lim_{c\to+\infty} (\alpha(\rho_0) - \int_{\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho)) = 0.$$

Let $c^* > c_*$ be such that $\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} \alpha(\rho)\mu_0(d\rho) < \varepsilon$. By lemma 3 below there exists sequence $\{\mu_n\}$ of measures in $\mathcal{P}^{\mathrm{f}}_{\{\rho_0\}}$ weakly converging to the measure μ_0 such that $\mu_n(\mathcal{A}_{c^*}) = \mu_0(\mathcal{A}_{c^*})$ and $\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} \alpha(\rho)\mu_n(d\rho) < \varepsilon$ for all n. Since the function f is upper semicontinuous and bounded on the set $\mathcal{A}_{c^*}$ we have (see [3])

$$\limsup_{n\to+\infty} \int_{\mathcal{A}_{c^*}} f(\rho)\mu_n(d\rho) \leq \int_{\mathcal{A}_{c^*}} f(\rho)\mu_0(d\rho).$$

Hence by noting that

$$\int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} f(\rho)\mu_n(d\rho) \leq M \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_{c^*}} \alpha(\rho)\mu_n(d\rho) < M\varepsilon, \quad n = 0, 1, 2...,$$

we obtain

$$\mathrm{co}f(\rho_0) \leq \liminf_{n\to+\infty} \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_n(d\rho) \leq \limsup_{n\to+\infty} \int_{\mathcal{A}_{c^*}} f(\rho)\mu_n(d\rho) + M\varepsilon$$

$$\leq \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu_0(d\rho) + M\varepsilon \leq \mu\text{-}\mathrm{co}f(\rho_0) + \varepsilon(M+1).$$

Since ε is arbitrary this implies $\mathrm{co}f(\rho_0) = \mu\text{-}\mathrm{co}f(\rho_0)$.

The proof of the first assertion of the theorem is completed by applying lemma 4 below.

By proposition 1 the second assertion of the theorem follows from the first one. $\square$

Lemma 3. *Let α be a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ with the range $[0, +\infty]$ and μ_0 be an arbitrary measure in $\mathcal{P}$. For given arbitrary $c > 0$ there exists a sequence $\{\mu_n\}$ of measures in $\mathcal{P}^{\mathrm{f}}_{\{\bar{\rho}(\mu_0)\}}$ converging to the measure μ_0 such that*

$$\mu_n(\mathcal{A}_c) = \mu_0(\mathcal{A}_c) \quad \textit{and} \quad \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_n(d\rho) = \int_{\mathfrak{S}(\mathcal{H})\setminus\mathcal{A}_c} \alpha(\rho)\mu_0(d\rho)$$

for all n, where $\mathcal{A}_c$ is the subset of $\mathfrak{S}(\mathcal{H})$ defined by (12).

Proof. This lemma can be proved by the simple modification of the proof of lemma 1 in [11], consisting in finding for given n of such decomposition of the set $\mathfrak{S}(\mathcal{H})$ into collection $\{\mathcal{A}_i^n\}_{i=1}^{m+2}$ of $m+2$ ($m = m(n)$) disjoint Borel subsets that

- the set $\mathcal{A}_i^n$ has diameter $< 1/n$ for $i = \overline{1, m}$;
- $\mu_0(\mathcal{A}_{m+1}^n) < 1/n$ and $\mu_0(\mathcal{A}_{m+2}^n) < 1/n$;
- the set $\mathcal{A}_i^n$ is contained either in $\mathcal{A}_c$ or in $\mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c$ for $i = \overline{1, m+2}$.

The essential point in this construction is the following implication

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow (\mu_0(\mathcal{A}))^{-1} \int_{\mathcal{A}} \rho\mu_0(d\rho) \in \mathcal{B}, \quad \text{where} \quad \mathcal{B} = \mathcal{A}_c,\ \mathfrak{S}(\mathcal{H}) \setminus \mathcal{A}_c,$$

and the equality

$$\int_{\mathcal{A}} \alpha(\rho)\mu_0(d\rho) = \mu_0(\mathcal{A})\,\alpha\left(\frac{1}{\mu_0(\mathcal{A})}\int_{\mathcal{A}} \rho\mu_0(d\rho)\right), \quad \mathcal{A} \subseteq \mathfrak{S}(\mathcal{H}),\ \mu_0(\mathcal{A}) \neq 0,$$

easily proved by using corollary A in the Appendix. $\square$

The contribution of property B of the set $\mathfrak{S}(\mathcal{H})$ to the proof of the above theorem is based on the following lemma.

Lemma 4. *Let α be a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ with the range $[0,+\infty]$ and f be a function on the set $\mathfrak{S}(\mathcal{H})$, which has upper semicontinuous restriction to the set $\mathcal{A}_c$ defined by (12) for each $c \geq 0$, then the function $\mathrm{co}f$ has upper semicontinuous restriction to the set $\mathcal{A}_c$ for each $c \geq 0$.*

Proof. Let $\rho_0 \in \mathcal{A}_{c_0}$ and let $\{\rho_n\} \subset \mathcal{A}_{c_0}$ be an arbitrary sequence converging to the state ρ_0. Suppose there exists

$$\lim_{n\to+\infty} \mathrm{co}f(\rho_n) > \mathrm{co}f(\rho_0). \tag{14}$$

For given arbitrary $\varepsilon > 0$ let $\{\pi_i^0, \rho_i^0\}_{i=1}^m$ be such ensemble in $\mathcal{P}^{\mathrm{f}}_{\{\rho_0\}}$ that $\sum_{i=1}^m \pi_i^0 f(\rho_i^0) < \mathrm{co}f(\rho_0) + \varepsilon$. By property B of the set $\mathfrak{S}(\mathcal{H})$ (in the form of lemma 3 in [24]) there exists sequence $\{\{\pi_i^n, \rho_i^n\}_{i=1}^m\}_n$ of ensembles such that $\sum_{i=1}^m \pi_i^n \rho_i^n = \rho_n$ for all n, $\lim_{n\to+\infty} \pi_i^n = \pi_i^0$ and $\lim_{n\to+\infty} \rho_i^n = \rho_i^0$ for all $i = \overline{1,m}$. Let $\pi_* = \min_{1\leq i\leq m} \pi_i^0$. Then there exists such N that $\pi_i^n \geq \pi_*/2$ for all $n \geq N$. It follows from inequality $\sum_{i=1}^m \pi_i^n \alpha(\rho_i^n) = \alpha(\rho_n) \leq c_0$ that $\rho_i^n \in \mathcal{A}_{\frac{2c_0}{\pi_*}}$ for all $n \geq N$ and $i = \overline{1,m}$. By upper semicontinuity of the function f on the set $\mathcal{A}_{\frac{2c_0}{\pi_*}}$ we have

$$\limsup_{n\to+\infty} \mathrm{co}f(\rho_n) \leq \limsup_{n\to+\infty} \sum_{i=1}^m \pi_i^n f(\rho_i^n) \leq \sum_{i=1}^m \pi_i^0 f(\rho_i^0) < \mathrm{co}f(\rho_0) + \varepsilon,$$

which contradicts to (14) since ε is arbitrary. $\square$

Remark 6. If f is a concave function then condition (13) follows from boundedness of the restriction of this function to the set $\mathcal{A}_c$ for each c. Indeed, for arbitrary affine function α concavity of the function f on the set $\mathfrak{S}(\mathcal{H})$ implies concavity of the function $c \mapsto \sup_{\rho\in\mathcal{A}_c} f(\rho)$ on the set $\mathbb{R}_+$, hence its finiteness guarantees validity of condition (13). If the function f has lower bounded upper semicontinuous restriction to the set $\mathcal{A}_c$ then by lemma 2 boundedness of this restriction follows from its finiteness at least at one state. $\square$

By this remark theorem 1 and proposition 5 imply the following result.

Corollary 5. *Let f be a concave lower semicontinuous lower bounded function and α be a lower semicontinuous affine function on the set $\mathfrak{S}(\mathcal{H})$*

with the range $[0,+\infty]$. If the function f has continuous restriction to the set $\mathcal{A}_c$ defined by (12) for each $c\geq 0$ then

$$\mathrm{co}f(\rho)=\sigma\text{-}\mathrm{co}f(\rho)=\mu\text{-}\mathrm{co}f(\rho)=\overline{\mathrm{co}}f(\rho)=(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})^{\sigma}_{*}(\rho)=(f|_{\mathrm{extr}\mathfrak{S}(\mathcal{H})})^{\mu}_{*}(\rho)$$

for all $\rho\in\bigcup_{c>0}\mathcal{A}_c$ and the common restriction of these functions to the set $\mathcal{A}_c$ is continuous for each $c\geq 0$.

Theorem 1 also implies the following two observations.

Corollary 6. *Let f be a Borel lower bounded function on the set $\mathfrak{S}(\mathcal{H})$ and ρ_0 be an arbitrary state in $\mathfrak{S}(\mathcal{H})$. If there exists affine lower semicontinuous function α on the set $\mathfrak{S}(\mathcal{H})$ with the range $[0,+\infty]$ such that $\alpha(\rho_0)<+\infty$, the function f has upper semicontinuous bounded restriction to the set $\mathcal{A}_c$ defined by (12) for each $c\geq 0$ and condition (13) holds then*

$$\mathrm{co}f(\rho_0)=\sigma\text{-}\mathrm{co}f(\rho_0)=\mu\text{-}\mathrm{co}f(\rho_0).$$

Corollary 7. *Let f be a lower semicontinuous lower bounded function on the set $\mathfrak{S}(\mathcal{H})$ and $\{\rho_n\}$ be an arbitrary sequence of states in $\mathfrak{S}(\mathcal{H})$ converging to the state ρ_0. If there exists affine lower semicontinuous function α on the set $\mathfrak{S}(\mathcal{H})$ with the range $[0,+\infty]$ such that $\sup_n\alpha(\rho_n)<+\infty$, the function f has continuous bounded restriction to the set $\mathcal{A}_c$ defined by (12) for each $c\geq 0$ and condition (13) holds then*

$$\mathrm{co}f(\rho_n)=\sigma\text{-}\mathrm{co}f(\rho_n)=\mu\text{-}\mathrm{co}f(\rho_n)=\overline{\mathrm{co}}f(\rho_n),\quad n=0,1,2...,\tag{15}$$

and

$$\lim_{n\to+\infty}\mathrm{co}f(\rho_n)=\mathrm{co}f(\rho_0).\tag{16}$$

Remark 7. If f is a concave function then condition (13) in corollaries 6 and 7 can be replaced by the condition $\sup_{\rho\in\mathcal{A}_c}f(\rho)<+\infty$ for all $c>0$ by remark 6. $\square$

Example 4. Let $\Phi:\mathfrak{T}(\mathcal{H})\mapsto\mathfrak{T}(\mathcal{H}')$ be an arbitrary quantum channel and $\mathfrak{S}(\mathcal{H})\ni\rho\mapsto(R_p\circ\Phi)(\rho)=\dfrac{\log\mathrm{Tr}\Phi(\rho)^p}{1-p}$ be the output Renyi entropy of this channel of order $p\in(0,+\infty]$ (the case $p=1$ corresponds to the output von Neumann entropy $-\mathrm{Tr}\Phi(\rho)\log\Phi(\rho)$, the case $p=+\infty$ corresponds to the function $-\log\lambda_{\max}(\Phi(\rho))$, where $\lambda_{\max}(\Phi(\rho))$ is the maximal eigenvalue of the state $\Phi(\rho)$). For $p\in(0,1]$ the function $R_p\circ\Phi$ is lower semicontinuous

concave and takes values in $[0, +\infty]$, while for $p \in (1, +\infty]$ it is continuous and finite but not concave. The output von Neumann entropy $H \circ \Phi = R_1 \circ \Phi$ is the supremum (pointwise limit as $p \to 1+0$) of the monotonous family of functions $\{R_p \circ \Phi\}_{p>1}$. By proposition 6 the convex closure $\overline{\mathrm{co}}(H \circ \Phi)$ of the output von Neumann entropy coincides with the supremum (pointwise limit as $p \to 1+0$) of the monotonous family of functions $\{\overline{\mathrm{co}}(R_p \circ \Phi)\}_{p>1}$.

Corollary 6 makes possible to show that

$$\mathrm{co}(R_p \circ \Phi)(\rho_0) = \sigma\text{-}\mathrm{co}(R_p \circ \Phi)(\rho_0) = \mu\text{-}\mathrm{co}(R_p \circ \Phi)(\rho_0) = \overline{\mathrm{co}}(R_p \circ \Phi)(\rho_0) \qquad (17)$$

for any state ρ_0 such that $(H \circ \Phi)(\rho_0) < +\infty$ all $p \in [1, +\infty]$. Indeed the condition $H(\Phi(\rho_0)) < +\infty$ implies existence of such $\mathfrak{H}$-operator H' in the space $\mathcal{H}'$ that $\mathrm{Tr}\exp(-\lambda H') < +\infty$ with $\lambda = 1$ and $\mathrm{Tr} H' \Phi(\rho_0) < +\infty$. By proposition 1 in [26] the conditions of corollary 6 are fulfilled for the function $f(\rho) = (R_p \circ \Phi)(\rho) \leq (H \circ \Phi)(\rho)$, $p \in [1, +\infty]$, with the affine function $\alpha(\rho) = \mathrm{Tr} H' \Phi(\rho)$. Note that if $(H \circ \Phi)(\rho_0) = +\infty$ then (17) may not be valid (see lemma 2 in [25]).

By corollary 1 the above coincidence of the convex hulls and continuity of the Renyi entropy for $p > 1$ imply continuity of the function $\mathrm{co}(R_p \circ \Phi)$ for $p > 1$ on the convex subset $\{\rho \in \mathfrak{S}(\mathcal{H}) \,|\, (H \circ \Phi)(\rho_0) < +\infty\}$.

If the output von Neumann entropy $H \circ \Phi$ is continuous on a particular subset $\mathcal{A} \subseteq \mathfrak{S}(\mathcal{H})$ then by proposition 7 in [25] its convex closure $\overline{\mathrm{co}}(H \circ \Phi)$ is also continuous and coincides with the convex hull $\mathrm{co}(H \circ \Phi)$ on this set. If the set $\mathcal{A}$ is compact then the above assertion on continuity of the function $\mathrm{co}(R_p \circ \Phi)$ and Dini's lemma imply uniform convergence of the function $\mathrm{co}(R_p \circ \Phi)|_{\mathcal{A}} = \overline{\mathrm{co}}(R_p \circ \Phi)|_{\mathcal{A}}$ to the function $\mathrm{co}(H \circ \Phi)|_{\mathcal{A}} = \overline{\mathrm{co}}(H \circ \Phi)|_{\mathcal{A}}$ as $p \to 1+0$. This shows in particular that the Holevo capacity of the $\mathcal{A}$-constrained channel Φ (see [11]) can be determined by the expression

$$\overline{C}(\Phi, \mathcal{A}) = \lim_{p \to 1+0} \sup_{\rho \in \mathcal{A}} \left((R_p \circ \Phi)(\rho) - \mathrm{co}(R_p \circ \Phi)(\rho) \right),$$

which can be used in analysis of continuity of the Holevo capacity as a function of a channel (since the Renyi entropy is continuous for $p > 1$).

5 Entanglement monotones

5.1 The basic properties

Entanglement is an essential feature of quantum systems, which can be considered as a special quantum correlation having no classical analogue. One of the basic tasks of the theory of entanglement consists in finding appropriate quantitative characteristics of entanglement of a state in composite system and in studying their properties (see [7],[19] and reference therein). In this section we consider infinite dimensional generalization of the "convex roof construction" of entanglement monotones and investigate its properties. This generalization is based on the results presented in the previous sections.

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. A state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ is called *separable* or *nonentangled* if it belongs to the convex closure of the set of all product pure states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, otherwise it is called *entangled*.

Entanglement monotone is an arbitrary nonnegative function E on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ having the following two properties (cf. [19],[32]):

EM-1) $\{E(\omega)=0\} \Leftrightarrow \{\textit{the state } \omega \textit{ is separable}\}$;

EM-2a) *Monotonicity of the function E under nonselective LOCC operations.* This means that

$$E(\omega) \geq E\left(\sum_{i,j} V_{ij}\omega V_{ij}^{*}\right) \tag{18}$$

for arbitrary state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and arbitrary LOCC protocol described by the Kraus operators $\{V_{ij}\}$.

EM-2b) *Monotonicity of the function E under selective LOCC operations.* This means that

$$E(\omega) \geq \sum_{i} \pi_i E(\omega_i), \quad \pi_i = \mathrm{Tr}\sum_{j} V_{ij}\omega V_{ij}^{*}, \quad \omega_i = \pi_i^{-1}\sum_{j} V_{ij}\omega V_{ij}^{*} \tag{19}$$

for arbitrary state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and arbitrary LOCC protocol described by the Kraus operators $\{V_{ij}\}$.

The natural generalization of the above requirement is the following.

EM-2c) *Monotonicity of the function E under generalized selective LOCC operations.* This means that for arbitrary state $\omega \in \mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and arbitrary local instrument $\mathfrak{M}$ with set of outcomes $\mathcal{X}$ the function $x \mapsto E(\sigma(x|\,\omega))$ is

μ_ω-measurable on the set $\mathcal{X}$ and

$$E(\omega) \geq \int_{\mathcal{X}} E(\sigma(x|\,\omega))\mu_\omega(dx), \tag{20}$$

where $\mu_\omega(\cdot) = \mathrm{Tr}\,\mathfrak{M}(\cdot)(\omega)$ and $\{\sigma(x|\,\omega)\}_{x\in\mathcal{X}}$ are respectively the probability measure on the set $\mathcal{X}$ describing the results of the measurement and the family of posteriori states corresponding to the a priori state ω [9],[16].

Remark 8. By definition the function $x \mapsto \sigma(x|\,\omega)$ is μ_ω-measurable with respect to the minimal σ-algebra on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ for which all linear functionals $\omega \mapsto \mathrm{Tr}A\omega,\ A \in \mathfrak{B}(\mathcal{H}\otimes\mathcal{K})$, are measurable. By corollary 1 in [31] this σ-algebra coincides with the Borel σ-algebra on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. Thus the function $x \mapsto \sigma(x|\,\omega)$ is μ_ω-measurable with respect to the Borel σ-algebra on $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ and hence the function $x \mapsto E(\sigma(x|\,\omega))$ is μ_ω-measurable for arbitrary Borel function $\omega \mapsto E(\omega)$.$\square$

According to [19] an entanglement monotone E is called *entanglement measure* if $E(\omega) = H(\mathrm{Tr}_{\mathcal{K}}\omega)$ for any pure state ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, where H is the von Neumann entropy.

Sometimes the following requirement is included in the definition of entanglement monotone (cf. [7]).

EM-3a) *Convexity of the function E on the set* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, which means that

$$E\left(\sum_i \pi_i\omega_i\right) \leq \sum_i \pi_i E(\omega_i)$$

for any finite ensemble $\{\pi_i, \omega_i\}$ of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. This requirement is due to the observation that entanglement can not be increased by taking convex mixtures.

The following two stronger forms of the convexity requirement are motivated by necessity to consider countable and continuous ensembles of states dealing with infinite dimensional quantum systems (cf. [11]).

EM-3b) *σ-convexity of the function E on the set* $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, which means that

$$E\left(\sum_i \pi_i\omega_i\right) \leq \sum_i \pi_i E(\omega_i)$$

for any countable ensemble $\{\pi_i, \omega_i\}$ of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. This requirement implies that EM-2b guarantees EM-2a.

EM-3c) *μ-convexity of the function E on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$*, which means that

$$E\left(\int_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})}\omega\mu(d\omega)\right)\leq\int_{\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})}E(\omega)\mu(d\omega)$$

for any Borel probability measure μ on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, which can be considered as a generalized (continuous) ensemble of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

In section 3 it is shown that these convexity properties are not equivalent in general. By Yensen's inequality (proposition A-2 in the Appendix) these properties are equivalent if the function E is either bounded and upper semicontinuous or lower semicontinuous (requirement EM-5a below).

EM-4) *Subadditivity of the function E*, which means that

$$E(\omega_1\otimes\omega_2)\leq E(\omega_1)+E(\omega_2) \tag{21}$$

for arbitrary states $\omega_1\in\mathfrak{S}(\mathcal{H}_1\otimes\mathcal{K}_1)$ and $\omega_2\in\mathfrak{S}(\mathcal{H}_2\otimes\mathcal{K}_2)$.

This property implies existence of the regularization

$$E^*(\omega)=\lim_{n\to+\infty}\frac{E(\omega^{\otimes n})}{n},\quad \omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}).$$

In the finite dimensional case it is natural to require continuity of entanglement monotone E on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$. In infinite dimensions this requirement is very restrictive. Moreover discontinuity of the von Neumann entropy implies discontinuity of any entanglement measure on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ in this case. Nevertheless some weaker continuity requirements may be considered.

EM-5a) *Lower semicontinuity of the function E on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.* This means that

$$\liminf_{n\to+\infty}E(\omega_n)\geq E(\omega_0)$$

for arbitrary sequence $\{\omega_n\}$ of states in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ converging to the state ω_0 or, equivalently, that the set of states defined by the inequality $E(\omega)\leq c$ is closed for any $c>0$. This requirement is motivated by the natural physical observation that entanglement can not be increased by an approximation procedure. It is essential that lower semicontinuity of the function E implies that this function is Borel and that requirements EM-3a – EM-3c are equivalent for this function (by proposition A-2 in the Appendix).

From the physical point of view it is natural to require that entanglement monotone is continuous on the set of states produced in a physical experiment. This leads to the following requirement.

EM-5b) *Continuity of the function E on subsets of $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ with bounded mean energy.* Let $H_{\mathcal{H}}$ and $H_{\mathcal{K}}$ be the Hamiltonians of the quantum systems associated with the spaces $\mathcal{H}$ and $\mathcal{K}$ correspondingly. Then the Hamiltonian of the composite system has the form $H_{\mathcal{H}}\otimes I_{\mathcal{K}}+I_{\mathcal{H}}\otimes H_{\mathcal{K}}$ and hence the set of states of the composite system with the mean energy not increasing h is defined by the inequality

$$\mathrm{Tr}(H_{\mathcal{H}}\otimes I_{\mathcal{K}}+I_{\mathcal{H}}\otimes H_{\mathcal{K}})\omega\leq h.$$

Requirement EM-5b means continuity of the restrictions of the function E to the subsets of $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ defined by the above inequality for all $h>0$.

The strongest continuity requirement is the following one.

EM-5c) *Continuity of the function E on the set $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.*

Despite infinite dimensionality there exists a nontrivial class of entanglement monotones for which this requirement holds (see example 5 in the next subsection.)

5.2 The generalized convex roof constructions

In the finite dimensional case a general way of producing of entanglement monotones is the "convex roof construction" (see [7, 15, 19]). By this construction for given concave continuous nonnegative function f on the set $\mathfrak{S}(\mathcal{H})$ such that

$$f^{-1}(0)=\mathrm{extr}\mathfrak{S}(\mathcal{H})\quad\text{and}\quad f(\rho)=f(U\rho U^{*})\tag{22}$$

for arbitrary state ρ in $\mathfrak{S}(\mathcal{H})$ and arbitrary unitary U in $\mathcal{H}$, the corresponding entanglement monotone E^{f} is defined as the convex roof $(f\circ\Theta|_{\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})})_{*}$ of the restriction of the function $f\circ\Theta$ to the set $\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$, where $\Theta:\omega\mapsto\mathrm{Tr}_{\mathcal{K}}\omega$ is a partial trace. By using the von Neumann entropy in the role of function f in the above construction we obtain the Entanglement of Formation E_{F} – one of the most important entanglement measures [2].

In the infinite dimensional case there exist two possible generalizations of the above construction: the σ-convex roof $(f\circ\Theta|_{\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})})_{*}^{\sigma}$ and the μ-convex roof $(f\circ\Theta|_{\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})})_{*}^{\mu}$ of the function $f\circ\Theta|_{\mathrm{extr}\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})}$. To simplify notations in what follows we will omit the symbol of restriction and will denote the above functions $(f\circ\Theta)_{*}^{\sigma}$ and $(f\circ\Theta)_{*}^{\mu}$ correspondingly.

The results of the previous sections make possible to prove the following observations concerning the main properties of these generalized convex roof constructions.

Theorem 2. *Let f be a nonnegative concave function on the set $\mathfrak{S}(\mathcal{H})$ satisfying condition (22).*

A-1) *If the function f is upper semicontinuous then*

$$(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}=\mu\text{-co}(f\circ\Theta)=\sigma\text{-co}(f\circ\Theta)=\mathrm{co}(f\circ\Theta),$$

this function is upper semicontinuous and satisfies requirements EM-1, EM-2c and EM-3c.

A-2) *If the function f is lower semicontinuous then the function $(f\circ\Theta)^{\sigma}_{*}$ satisfies requirements EM-2b and EM-3b while the function $(f\circ\Theta)^{\mu}_{*}$ coincides with the function $\overline{\mathrm{co}}(f\circ\Theta)$ and satisfies requirements EM-1, EM-2c, EM-3c and EM-5a.*[14]

B) *If the function f is subadditive*[15] *then the functions $(f\circ\Theta)^{\sigma}_{*}$ and $(f\circ\Theta)^{\mu}_{*}$ satisfy requirement EM-4.*

C) *Let $H_{\mathcal{H}}$ be a positive operator in the space $\mathcal{H}$. If the function f is lower semicontinuous and has finite continuous restriction to the subset $\mathcal{K}_{H_{\mathcal{H}},h}=\{\rho\in\mathfrak{S}(\mathcal{H})\,|\,\mathrm{Tr}H_{\mathcal{H}}\rho\leq h\}$ for each $h>0$ then*

$$(f\circ\Theta)^{\sigma}_{*}(\omega)=(f\circ\Theta)^{\mu}_{*}(\omega)=\overline{\mathrm{co}}(f\circ\Theta)(\omega)=\mathrm{co}(f\circ\Theta)(\omega),\quad \forall\omega\in\bigcup_{h>0}\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h},$$

where $\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h}=\{\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})\,|\,\mathrm{Tr}(H_{\mathcal{H}}\otimes I_{\mathcal{K}})\,\omega\leq h\}$, and the common restriction of these functions to the set $\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h}$ is continuous for each $h>0$. In particular, if $H_{\mathcal{H}}$ is the Hamiltonian of the quantum system associated with the space $\mathcal{H}$ then the functions $(f\circ\Theta)^{\mu}_{*}$ and $(f\circ\Theta)^{\sigma}_{*}$ satisfy requirement EM-5b.

D) *If the function f is continuous then*

$$(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}=\overline{\mathrm{co}}(f\circ\Theta)=\mathrm{co}(f\circ\Theta)$$

and the function $(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}$ satisfies requirement EM-5c.

Proof. By proposition 5, lemma 2 and proposition A-2 in the Appendix upper semicontinuity of the function f imply

$$(f\circ\Theta)^{\mu}_{*}=(f\circ\Theta)^{\sigma}_{*}=\mu\text{-co}(f\circ\Theta)=\sigma\text{-co}(f\circ\Theta)=\mathrm{co}(f\circ\Theta),$$

[14] The example in remark 9 below shows that the function $(f\circ\Theta)^{\sigma}_{*}$ may not satisfy requirements EM-1, EM-3c and EM-5a even for bounded lower semicontinuous function f.

[15] This means that $f(\rho_1\otimes\rho_2)\leq f(\rho_1)+f(\rho_2)$ for arbitrary $\rho_1\in\mathfrak{S}(\mathcal{H}_1)$ and $\rho_2\in\mathfrak{S}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable Hilbert spaces (we implicitly use the isomorphism between all such spaces).

validity of requirement EM-3c for the last function and its upper semicontinuity.

By proposition 3 lower semicontinuity of the function f implies lower semicontinuity of the function $(f\circ\Theta)^{\mu}_{*}$, t.i. validity of requirement EM-5a for this function. Hence proposition A-2 in the Appendix implies validity of requirement EM-3c for the function $(f\circ\Theta)^{\mu}_{*}$ in this case.

Validity of requirement EM-3b for the function $(f\circ\Theta)^{\sigma}_{*}$ follows from its definition.

By repeating the arguments used in the proof of LOCC monotonicity of the convex roof of the function $f\circ\Theta$ in the finite dimensional case (see [2],[19]) and by using proposition A-1 in the Appendix validity of requirement EM-2b for the function $(f\circ\Theta)^{\sigma}_{*}$ can be proved.

Consider requirement EM-2c. Let $\mathfrak{M}$ be an arbitrary instrument acting in the subsystem associated with the space $\mathcal{K}$. If the function f is lower (corresp. upper) semicontinuous then the function $(f\circ\Theta)^{\mu}_{*}$ is lower (corresp. upper) semicontinuous and hence it is Borel. By remark 8 this guarantees μ_{ω}-measurability of the function $x\mapsto E(\sigma(x|\,\omega))$ for arbitrary state ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$.

Let ω be a pure state. By locality of the instrument $\mathfrak{M}$ we have

$$\Theta(\omega)=\int_{\mathcal{X}}\Theta(\sigma(x|\,\omega))\mu_{\omega}(dx)$$

Since f is nonnegative concave lower semicontinuous or upper semicontinuous function, proposition A-2 in the Appendix implies

$$f\circ\Theta(\omega)\geq\int_{\mathcal{X}}f\circ\Theta(\sigma(x|\,\omega))\mu_{\omega}(dx)\geq\int_{\mathcal{X}}(f\circ\Theta)^{\mu}_{*}(\sigma(x|\,\omega))\mu_{\omega}(dx),$$

where the last inequality follows from proposition 5.

Let ω be a mixed state. Prove first that

$$(f\circ\Theta)^{\sigma}_{*}(\omega)\geq\int_{\mathcal{X}}(f\circ\Theta)^{\mu}_{*}(\sigma(x|\,\omega))\mu_{\omega}(dx). \tag{23}$$

For given $\varepsilon>0$ let $\{\pi_i,\omega_i\}$ be such ensemble in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ that

$$(f\circ\Theta)^{\sigma}_{*}(\omega)>\sum_i\pi_i f\circ\Theta(\omega_i)-\varepsilon.$$

By the observation concerning pure state ω we have

$$(f\circ\Theta)_*^\sigma(\omega)>\sum_i \pi_i\int_{\mathcal{X}}(f\circ\Theta)_*^\mu(\sigma(x|\,\omega_i))\mu_{\omega_i}(dx)-\varepsilon. \tag{24}$$

By the Radon-Nicodym theorem the decomposition

$$\mu_\omega(\cdot)=\mathrm{Tr}\,\mathfrak{M}(\cdot)(\omega)=\sum_i \pi_i \mathrm{Tr}\,\mathfrak{M}(\cdot)(\omega_i)=\sum_i \pi_i\mu_{\omega_i}(\cdot)$$

implies existence of family $\{p_i\}$ of μ_ω-measurable functions on $\mathcal{X}$ such that

$$\pi_i\mu_{\omega_i}(\mathcal{X}_0)=\int_{\mathcal{X}_0}p_i(x)\mu_\omega(dx)$$

for arbitrary μ_ω-measurable subset $\mathcal{X}_0\subseteq\mathcal{X}$ and $\sum_i p_i(x)=1$ for μ_ω-almost all x in $\mathcal{X}$. Since

$$\int_{\mathcal{X}_0}\sigma(x|\,\omega)\mu_\omega(dx)=\sum_i\pi_i\int_{\mathcal{X}_0}\sigma(x|\,\omega_i)\mu_{\omega_i}(dx)=\sum_i\int_{\mathcal{X}_0}\sigma(x|\,\omega_i)p_i(x)\mu_\omega(dx)$$

for arbitrary μ_ω-measurable subset $\mathcal{X}_0\subseteq\mathcal{X}$ we have

$$\sum_i p_i(x)\sigma(x|\,\omega_i)=\sigma(x|\,\omega)$$

for μ_ω-almost all x in $\mathcal{X}$.

Note that the function $(f\circ\Theta)_*^\mu$ is σ-convex in the both cases. Indeed, if f is an upper semicontinuous function this follows from its coincidence with the function $(f\circ\Theta)_*^\sigma$, if f is a lower semicontinuous function then the convex function $(f\circ\Theta)_*^\mu$ is lower semicontinuous and hence μ-convex (by proposition A-2 in the Appendix).

By using (24) and σ-convexity of the function $(f\circ\Theta)_*^\mu$ we obtain

$$(f\circ\Theta)_*^\sigma(\omega)>\int_{\mathcal{X}}\sum_i p_i(x)(f\circ\Theta)_*^\mu(\sigma(x|\,\omega_i))\mu_\omega(dx)-\varepsilon$$

$$\geq\int_{\mathcal{X}}(f\circ\Theta)_*^\mu(\sigma(x|\,\omega))\mu_\omega(dx)-\varepsilon$$

which implies (23) since ε is arbitrary.

If f is an upper semicontinuous function then $(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}$ and (23) means (20) for the function $E=(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}$.

If f is a lower semicontinuous function then for a given arbitrary state $\omega\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ lemma 1 and proposition 5 imply existence of a sequence $\{\omega_n\}\subset\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ converging to the state ω such that

$$\lim_{n\to+\infty}(f\circ\Theta)^{\sigma}_{*}(\omega_n)=(f\circ\Theta)^{\mu}_{*}(\omega).$$

Inequality (20) for the function $E=(f\circ\Theta)^{\mu}_{*}$ can be proved by applying inequality (23) for each state in the sequence $\{\omega_n\}$ and passing to the limit $n\to+\infty$ by means of lemma A-2 in the Appendix and due to lower semicontinuity of the function $(f\circ\Theta)^{\mu}_{*}$.

Consider requirement EM-1. Note that a state ω is separable if and only if there exists a measure μ in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ supported by pure product states [12].

Let f be a lower semicontinuous function. By proposition 3 for arbitrary state ω in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ there exists a measure μ_{ω} in $\widehat{\mathcal{P}}_{\{\omega\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ such that $(f\circ\Theta)^{\mu}_{*}(\omega)=\int f\circ\Theta(\sigma)\mu_{\omega}(d\sigma)$. Hence validity of requirement EM-1 for the function $(f\circ\Theta)^{\mu}_{*}$ follows from the above characterization of the set of separable states.

Let f be an upper semicontinuous function. Then the function $(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}$ equals to zero on the set of separable states by the above characterization of this set.

Suppose this function equals to zero at some entangled state ω_0. Then there exists local operation Λ such that the state $\Lambda(\omega_0)$ is entangled and has reduced states of finite rank. By LOCC monotonicity of the function $(f\circ\Theta)^{\sigma}_{*}=(f\circ\Theta)^{\mu}_{*}$ proved before this function equals to zero at the entangled state $\Lambda(\omega_0)$.

Let $\mathcal{H}_0$ be the finite dimensional support of the state $\mathrm{Tr}_{\mathcal{K}}\Lambda(\omega_0)$. Then upper semicontinuous concave function f satisfying condition (22) has continuous restriction to the set $\mathfrak{S}(\mathcal{H}_0)$. Indeed, continuity of this restriction at any pure state in $\mathfrak{S}(\mathcal{H}_0)$ follows from upper semicontinuity of the nonnegative function f and condition (22) while continuity of this restriction at any mixed state in $\mathfrak{S}(\mathcal{H}_0)$ can be easily derived from the well known fact that any concave bounded function is continuous at any internal point of a convex subset of a Banach space (proposition 3.2.3 in [14]). Since

$$(f\circ\Theta|_{\mathfrak{S}(\mathcal{H}_0\otimes\mathcal{K})})^{\mu}_{*}=(f\circ\Theta)^{\mu}_{*}|_{\mathfrak{S}(\mathcal{H}_0\otimes\mathcal{K})}$$

we can apply the previous observation concerning lower semicontinuous function f to show that equality $(f\circ\Theta)^{\mu}_{*}(\Lambda(\omega_0))=0$ implies separability of the state $\Lambda(\omega_0)$, contradicting to the above assumption.

B) If the function f is subadditive then the function $f\circ\Theta$ is subadditive as well. Let $\mu_i\in\widehat{\mathcal{P}}_{\{\omega_i\}}(\mathfrak{S}(\mathcal{L}_i))$, where $\mathcal{L}_i=\mathcal{H}_i\otimes\mathcal{K}_i$, $i=1,2$, be arbitrary measures. The set of product states in $\mathrm{extr}\mathfrak{S}(\mathcal{L}_1\otimes\mathcal{L}_2)$ can be considered as the Cartesian product of the sets $\mathrm{extr}\mathfrak{S}(\mathcal{L}_1)$ and $\mathrm{extr}\mathfrak{S}(\mathcal{L}_2)$. Hence on this set one can define the Cartesian product of the measures μ_1 and μ_2, denoted by $\mu_1\otimes\mu_2$, which can be considered as a measure in $\widehat{\mathcal{P}}_{\{\omega_1\otimes\omega_2\}}(\mathfrak{S}(\mathcal{L}_1\otimes\mathcal{L}_2))$ supported by the set of product states. By using this construction it is easy to prove subadditivity of the function $(f\circ\Theta)^{\mu}_{*}$. By the same arguments with atomic measures μ_1 and μ_2 one can prove subadditivity of the function $(f\circ\Theta)^{\sigma}_{*}$.[16]

C) If the function f is lower semicontinuous and satisfies the additional conditions in C, then the function $f\circ\Theta$ satisfies the conditions of corollary 5 with the affine function $\alpha(\omega)=\mathrm{Tr}(H_{\mathcal{H}}\otimes I_{\mathcal{K}})\omega$.

D) Assertion D follows from proposition 5. $\square$

Remark 9. The function $(f\circ\Theta)^{\sigma}_{*}$ may not satisfy the basic requirement EM-1 even for bounded lower semicontinuous function f (assertion A-2). Indeed, let f be the indicator function of the set of all mixed states in $\mathfrak{S}(\mathcal{H})$ and ω_0 be such separable state that any measure in $\widehat{\mathcal{P}}_{\{\omega_0\}}(\mathfrak{S}(\mathcal{H}\otimes\mathcal{K}))$ has no atoms in the set of separable states [12]. Then it is easy to see that $(f\circ\Theta)^{\sigma}_{*}(\omega_0)=1$ (while $(f\circ\Theta)^{\mu}_{*}(\omega_0)=0$!).

The function $(f\circ\Theta)^{\sigma}_{*}$ in the above example does not also satisfy requirements EM-3c and EM-5a. This is a general feature of any σ-convex roof not coinciding with the corresponding μ-convex roof. $\square$

The above remark and the assertions of theorem 2 show that the function $(f\circ\Theta)^{\sigma}_{*}$ either coincides with the function $(f\circ\Theta)^{\mu}_{*}$ (if f is upper semicontinuous) or may not satisfy the basic requirement EM-1 of entanglement monotone (if f is lower semicontinuous). Thus the μ-convex roof construction seems to be more *preferable* candidate on the role of infinite dimensional generalization of the convex roof construction of entanglement monotones. Thus we will use the following notation

$$E^{f}=(f\circ\Theta)^{\mu}_{*}$$

[16] In this case the measure $\mu_1\otimes\mu_2$ corresponds to the tensor product of countable ensembles of pure states corresponding to the measures μ_1 and μ_2.

for arbitrary function f satisfying the conditions of theorem 2.

Example 5. Generalizing to the infinite dimensional case the observation in [15] consider the family of functions

$$f_\alpha(\rho) = 2(1 - \mathrm{Tr}\rho^\alpha), \quad \alpha > 1,$$

on the set $\mathfrak{S}(\mathcal{H})$ with $\dim \mathcal{H} = +\infty$. The functions of this family are nonnegative concave continuous and satisfy conditions (22). By theorem 2 E^{f_α} is an entanglement monotone, satisfying requirements EM-1, EM-2c, EM-3c and EM-5c. In the case $\alpha = 2$ the entanglement monotone E^{f_2} can be considered as the infinite dimensional generalization of the I-tangle [23]. By corollary 4 the function $(\omega, \alpha) \mapsto E^{f_\alpha}(\omega)$ is continuous on the set $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \times [1, +\infty)$. By corollary 3 the least upper bound of the monotonous family $\{E^{f_\alpha}\}_{\alpha>1}$ of continuous entanglement monotones coincides with the characteristic function of the set of entangled states. $\square$

Example 6. Let $R_p(\rho) = \dfrac{\log \mathrm{Tr}\rho^p}{1-p}$ be the Renyi entropy of order $p \in [0, 1]$ (the case $p = 0$ corresponds to the function $\log \mathrm{rank}(\rho)$, the case $p = 1$ corresponds to the von Neumann entropy $-\mathrm{Tr}\rho \log \rho$).

R_p is a concave lower semicontinuous subadditive function with the range $[0, +\infty]$, satisfying condition (22). By theorem 2 the function E^{R_p} is an entanglement monotone, satisfying requirements EM-1, EM-2c, EM-3c, EM-4 and EM-5a. In the case $p = 0$ the entanglement monotone E^{R_0} is an infinite dimensional generalization of the Schmidt measure [7]. In the case $p = 1$ the entanglement monotone $E^{R_1} = E^H$ is an entanglement measure, which can be considered as an infinite dimensional generalization of the Entanglement of Formation [2] (see section 6). If $\inf\{\lambda > 0 \,|\, \mathrm{Tr}\exp(-\lambda H_{\mathcal{H}}) < +\infty\} = 0$ then theorem 2C implies that the entanglement measure $E^{R_1} = E^H$ satisfies requirement EM-5b since the von Neumann entropy $H = R_1$ is continuous on the set $\mathcal{K}_{H_{\mathcal{H}},h}$ (see the observation in [33] or proposition 1a in [26]). The last assertion was originally proved in [25] as a corollary of the general continuity condition for the function $E^H = (H \circ \Theta)^\mu_* = \overline{\mathrm{co}}(H \circ \Theta)$ obtained by using the special relation between the von Neumann entropy and the relative entropy.

According to [34] the entanglement monotones of the family $\{E^{R_p}\}_{p\in[0,1]}$ can be called Generalized Entanglement of Formation. $\square$

5.3 Approximation

In general entanglement monotones produced by the μ-convex roof construction are unbounded and discontinuous (only lower or upper semicontinuous), which may lead to analytical problems in dealing with these functions. Some of these problems can be solved by using the following approximation result.

Proposition 7. *Let f be a concave nonnegative lower semicontinuous (correspondingly upper semicontinuous) function on the set $\mathfrak{S}(\mathcal{H})$ satisfying condition (22), which is represented as a limit of increasing (correspondingly decreasing) sequence $\{f_n\}$ of concave continuous nonnegative functions on the set $\mathfrak{S}(\mathcal{H})$ satisfying condition (22). Then the entanglement monotone E^f is a limit of the increasing (correspondingly decreasing) sequence $\{E^{f_n}\}_n$ of continuous entanglement monotones.*

If in addition the function f satisfies condition C in theorem 2 then the sequence $\{E^{f_n}\}$ converges to the entanglement monotone E^f uniformly on compact subsets of the set $\mathcal{K}_{H_{\mathcal{H}}\otimes I_{\mathcal{K}},h}$ for each $h > 0$.

Poof. The first assertion of this proposition follows from theorem 2, corollary 3 and remark 5. The second assertion follows from the first one and Dini's lemma. $\square$

6 Entanglement of Formation

6.1 The definitions

The Entanglement of Formation of a state ω of a finite dimensional composite system is defined in [2] as the minimal possible average entanglement over all pure state *discrete finite* decompositions of ω (entanglement of pure state is defined as the von Neumann entropy of a reduced state). In our notations this means that

$$E_F = (H \circ \Theta)_* = \overline{\mathrm{co}}(H \circ \Theta) = \mathrm{co}(H \circ \Theta).$$

The possible generalization of this notion is considered in [6], where the Entanglement of Formation of a state ω of an infinite dimensional composite system is defined as the minimal possible average entanglement over all pure state *discrete countable* decompositions of ω, which means $E_F^d = (H \circ \Theta)_*^\sigma$.

The generalized convex roof construction described above with the von Neumann entropy H in the role of function f leads to the definition of EoF

as $E_F^c = E^H = (H \circ \Theta)_*^\mu = \overline{\mathrm{co}}(H \circ \Theta)$ considered in [25][17], by which the Entanglement of Formation of a state ω of an infinite dimensional composite system is defined as the minimal possible average entanglement over all pure state *continuous* decompositions of ω.

An interesting open question is a relation between E_F^d and E_F^c. It follows from the definitions that

$$E_F^d(\omega) \geq E_F^c(\omega), \quad \forall \omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}).$$

In [25] it is shown that

$$E_F^d(\omega) = E_F^c(\omega) \tag{25}$$

for arbitrary state ω such that either $H(\mathrm{Tr}_{\mathcal{H}}\omega) < +\infty$ or $H(\mathrm{Tr}_{\mathcal{K}}\omega) < +\infty$. Equality (25) obviously holds for all pure states and for all nonentangled states, but its validity for arbitrary state ω is not proved (up to my knowledge). Example in remark 9 shows that this question can not be solved by using only such analytical properties of the von Neumann entropy as concavity and lower semicontinuity. Note that the question of coincidence of the functions E_F^d and E_F^c is equivalent to the question of lower semicontinuity of the function E_F^d since E_F^c is the greatest lower semicontinuous convex function coinciding with the von Neumann entropy on the set of pure states.

Despite the fact that the definition of the function E_F^d seems more reasonable from the physical point of view (since it involves optimization over ensembles of quantum states rather then measures) the assumption of existence of a state ω_0 such that $E_F^d(\omega_0) \neq E_F^c(\omega_0)$ leads to the following "nonphysical" property of the function E_F^d. For each natural n consider local measurement $\{M_k^n\}_{k\in\mathbb{N}}$, where

$$M_1 = \left(\sum_{i=1}^{n} |i\rangle\langle i|\right) \otimes I_{\mathcal{K}} \quad \text{and} \quad M_k = |n+k-1\rangle\langle n+k-1| \otimes I_{\mathcal{K}}, \quad k > 1.$$

It is clear that the sequence $\{\Phi_n = \{M_k^n\}_{k\in\mathbb{N}}\}_n$ of nonselective local measurements tends to the trivial measurement – identity transformation. Since the functions E_F^d and E_F^c satisfy requirement EM-2b and EM-3b we have

$$E_F^d(\omega_0) \geq \sum_{k=1}^{+\infty} \pi_k E_F^d(\omega_k) \geq E_F^d\left(\sum_{k=1}^{+\infty} \pi_k \omega_k\right) = E_F^d(\Phi_n(\omega_0))$$

[17]In this paper the functions E_F^d and E_F^c are denoted E_F^1 and E_F^2 correspondingly.

and

$$E_F^c(\omega_0) \geq \sum_{k=1}^{+\infty} \pi_k E_F^c(\omega_k),$$

where ω_k is the posteriori state with the outcome k and π_k is the probability of this outcome.

Since for each k the state $\mathrm{Tr}_{\mathcal{K}}\omega_k$ has finite rank we have $E_F^d(\omega_k) = E_F^c(\omega_k)$ (by the result in [25] mentioned before). Thus the above inequalities imply

$$E_F^d(\Phi_n(\omega_0)) = E_F^d\left(\sum_{k=1}^{+\infty} \pi_k \omega_k\right) \leq E_F^c(\omega_0)$$

for all n and hence

$$\limsup_{n\to+\infty} E_F^d(\Phi_n(\omega_0)) \leq E_F^d(\omega_0) - \Delta, \quad \text{where} \quad \Delta = E_F^d(\omega_0) - E_F^c(\omega_0) > 0,$$

despite the fact that the sequence $\{\Phi_n\}_n$ of nonselective local measurements tends to the identity transformation. In contrast to this lower semicontinuity of the function E_F^c implies

$$\lim_{n\to+\infty} E_F^c(\Phi_n(\omega_0)) = E_F^c(\omega_0)$$

for arbitrary state ω_0 and arbitrary sequence $\{\Phi_n\}_n$ of nonselective LOCC-operations tending to the identity transformation.

The another advantage of the function E_F^c consists in validity of requirements EM-2c for this function while the assumption $E_F^d \neq E_F^c$ means that the function E_F^d is not lower semicontinuous, which is a real obstacle to prove the analogous property for this function.

6.2 The approximation by continuous entanglement monotones

For given natural $n > 1$ consider the function

$$\mathfrak{S}(\mathcal{H}) \ni \rho \mapsto H_n(\rho) = \sup \sum_i \pi_i H(\rho_i),$$

where the supremum is over all countable ensembles $\{\pi_i, \rho_i\}$ of states of rank $\leq n$ such that $\sum_i \pi_i \rho_i = \rho$. It is easy to see that the function H_n is concave

and that this function satisfies condition (22), has the range $[0, \log n]$ and coincides with the Neumann entropy on the subset of $\mathfrak{S}(\mathcal{H})$ consisting of states of rank $\leq n$. By using the strengthen version of property A of the set $\mathfrak{S}(\mathcal{H})$ it is shown in [29] that this function is continuous on the set $\mathfrak{S}(\mathcal{H})$ and that the increasing sequence $\{H_n\}$ pointwise converges to the von Neumann entropy on this set.

By theorem 2 the function $E_F^n = (H_n \circ \Theta)_*^\mu = (H_n \circ \Theta)_*^\sigma = \mathrm{co}(H_n \circ \Theta)$ is an entanglement monotone satisfying requirements EM-1, EM-2c, EM-3c, EM-4 and EM-5c. It is easy to see that this function has the range $[0, \log n]$ and coincides with the function E_F^c on the set

$$\{\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K}) \,|\, \min\{\mathrm{rank}\mathrm{Tr}_{\mathcal{K}}\omega, \mathrm{rank}\mathrm{Tr}_{\mathcal{H}}\omega\} \leq n\}\,.$$

By proposition 7 the sequence $\{E_F^n\}$ provides approximation of the function E_F^c on the set $\mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$, which is uniform on each compact set of continuity of the function E_F^c, in particular, on the set of states of Bosonic composite system with bounded mean energy.

6.3 The continuity conditions

Proposition 7 in [25] implies the following continuity condition for the function E_F^c, which can be also formulated as a continuity condition for the function E_F^d since this condition implies coincidence of these functions.

Proposition 8. *The function E_F^c has continuous restriction to the set $\mathcal{A} \subset \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ if either the function $\omega \mapsto H(\mathrm{Tr}_{\mathcal{H}}\omega)$ or the function $\omega \mapsto H(\mathrm{Tr}_{\mathcal{K}}\omega)$ has continuous restriction to the set $\mathcal{A}$.*

This condition implies the result mentioned in example 6 (validity of requirement EM-5b) as well as the following observation.

Corollary 8. *The function E_F^c has continuous restriction to the set $\{\omega \,|\, \mathrm{Tr}_{\mathcal{K}}\omega = \rho \in \mathfrak{S}(\mathcal{H})\}$ if and only if $H(\rho) < +\infty$.*

Proof. It is sufficient to note that if $H(\rho) = +\infty$ then there exists pure state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $\mathrm{Tr}_{\mathcal{K}}\omega = \rho$. $\square$

By corollary 8 for arbitrary continuous family $\{\Psi_t\}_t$ of local operations on the quantum system associated with the space $\mathcal{K}$ and arbitrary state $\omega \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{K})$ such that $\mathrm{Tr}_{\mathcal{K}}\omega < +\infty$ the function $t \mapsto E_F^c(\Psi_t(\omega))$ is continuous.

For arbitrary state σ let $\mathrm{dc}(\sigma) = \inf\{\lambda \in \mathbb{R} \mid \mathrm{Tr}\sigma^{\lambda} < +\infty\}$ be the characteristic of the spectrum of this state. It is clear that $\mathrm{dc}(\sigma) \in [0, 1]$. Proposition 8, proposition 2 in [26] and the monotonicity property of the relative entropy imply the following condition of continuity of the function E_F^c with respect to the convergence defined by the relative entropy (which is more stronger than the convergence defined by the trace norm).

Corollary 9. *Let ω_0 be such state in $\mathfrak{S}(\mathcal{H}\otimes\mathcal{K})$ that either $\mathrm{dc}(\mathrm{Tr}_{\mathcal{H}}\omega) < 1$ or $\mathrm{dc}(\mathrm{Tr}_{\mathcal{K}}\omega) < 1$. If $\{\omega_n\}$ is such sequence that $\lim_{n\to+\infty} H(\omega_n\|\omega_0) = 0$ then $\lim_{n\to+\infty} E_F^c(\omega_n) = E_F^c(\omega_0)$.*

7 Appendix

7.1 Yensen's inequalities

In contrast to the case of $\mathbb{R}^n$ for convex functions defined on convex subsets of separable Banach spaces the well known Yensen's inequality does not hold in general (see examples in section 3). Below the sufficient conditions of validity of this inequality in discrete and integral forms are presented.

Proposition A-1. (discrete Yensen's inequality) *Let f be a convex upper bounded function on closed convex subset $\mathcal{A}$ of a Banach space. Then for arbitrary countable set $\{x_i\} \subset \mathcal{A}$ with corresponding probability distribution $\{\pi_i\}$ the following inequality holds*

$$f\left(\sum_{i=1}^{+\infty} \pi_i x_i\right) \le \sum_{i=1}^{+\infty} \pi_i f(x_i). \tag{26}$$

Proof. Let $\bar{x} = \sum_{i=1}^{+\infty} \pi_i x_i$, $\lambda_n = \sum_{i=1}^{n} \pi_i$ and $\bar{x}_n = \lambda_n^{-1} \sum_{i=1}^{n} \pi_i x_i$. By convexity of the function f we have

$$\begin{aligned} f(\bar{x}) = f\left(\lambda_n \bar{x}_n + (1-\lambda_n)\frac{\bar{x}-\lambda_n\bar{x}_n}{1-\lambda_n}\right) &\le \lambda_n f(\bar{x}_n) + (1-\lambda_n) f\left(\frac{\bar{x}-\lambda_n\bar{x}_n}{1-\lambda_n}\right) \\ &\le \sum_{i=1}^{n} \pi_i f(x_i) + (1-\lambda_n) f\left(\frac{\bar{x}-\lambda_n\bar{x}_n}{1-\lambda_n}\right). \end{aligned}$$

By upper boundedness of the function f passing to the limit $n \to +\infty$ implies (26).$\square$

Proposition A-2. (integral Yensen's inequality) *Let f be a convex function on closed bounded convex subset $\mathcal{A}$ of a separable Banach space which is either lower semicontinuous or upper bounded upper semicontinuous. Then for arbitrary Borel probability measure μ on the set $\mathcal{A}$ the following inequality holds*

$$f\left(\int_{\mathcal{A}} x\mu(dx)\right) \leq \int_{\mathcal{A}} f(x)\mu(dx). \tag{27}$$

Proof. Let μ_0 be an arbitrary probability measure on the set $\mathcal{A}$.

Let f be an upper bounded upper semicontinuous function. Then the function $\mu \mapsto \int_{\mathcal{A}} f(x)\mu(dx)$ is upper semicontinuous on the set $\mathcal{P}(\mathcal{A})$ of Borel probability measures on the set $\mathcal{A}$ endowed with the weak topology [3],[18]. Let $\{\mu_n\}$ be a sequence of measures with finite support and the same barycenter as the measure μ_0 weakly converging to the measure μ_0. By convexity of the function f inequality (27) holds with $\mu = \mu_n$ for each n. By upper semicontinuity of the function $\mu \mapsto \int_{\mathcal{A}} f(x)\mu(dx)$ passing to the limit $n \to +\infty$ in this inequality implies inequality (27) with $\mu = \mu_0$.

Let f be a lower semicontinuous function. By using the arguments from the proof of lemma 2 one can show that the function f is either lower bounded or does not take finite values. It is sufficient to consider the first case. Suppose that $\int_{\mathcal{A}} f(x)\mu(dx) < +\infty$. By applying the construction used in the proof of lemma 1 it is possible to obtain sequence $\{\mu_n\}$ of measures on the set $\mathcal{A}$ with finite support such that

$$\limsup_{n \to +\infty} \int_{\mathcal{A}} f(x)\mu_n(dx) \leq \int_{\mathcal{A}} f(x)\mu_0(dx) \quad \text{and} \quad \lim_{n \to +\infty} \int_{\mathcal{A}} x\mu_n(dx) = \int_{\mathcal{A}} x\mu_0(dx).$$

By convexity of the function f inequality (27) holds with $\mu = \mu_n$ for each n. By lower semicontinuity of the function f passing to the limit $n \to +\infty$ implies inequality (27) with $\mu = \mu_0$. $\square$

Since any affine function is convex and concave simultaneously proposition A-2 and the arguments from the proof of lemma 2 imply the following result.

Corollary A. *Let f be an affine lower semicontinuous function on closed bounded convex subset $\mathcal{A}$ of a separable Banach space. Then for arbitrary Borel probability measure μ on the set $\mathcal{A}$ the following equality holds*

$$f\left(\int_{\mathcal{A}} x\mu(dx)\right) = \int_{\mathcal{A}} f(x)\mu(dx). \tag{28}$$

The simplest example of a Borel affine function for which (28) does not hold for some measures is the function on the simplex of all probability distributions with countable number of outcomes taking the value 0 on the convex set of distributions with finite nonzero elements and the value $+\infty$ on its complement.

7.2 The converse of proposition 6

Here the proof of the assertion in remark 3 is presented.

Proposition A-3. *Let $\mathcal{A}$ be a convex bounded closed subset of a separable Banach space and $\mathcal{P}(\mathcal{A})$ be the set of Borel probability measures on the set $\mathcal{A}$ endowed with the week convergence topology. If for arbitrary increasing sequence $\{f_n\}$ of concave continuous bounded functions on the set $\mathcal{A}$ with continuous bounded pointwise limit f_0 the sequence $\{\overline{\mathrm{co}} f_n\}$ pointwise converges to the function $\overline{\mathrm{co}} f_0$ then for arbitrary compact subset $\mathcal{K} \subseteq \mathcal{A}$ the set $b^{-1}(\mathcal{K})$ is a compact subset of $\mathcal{P}(\mathcal{A})$, where $b : \mathcal{P}(\mathcal{A}) \rightarrow \mathcal{A}$ is the barycenter map.*

Proof. Suppose the asserted property does not hold. This leads to the following two cases. The first case consists in existence of such $x_0 \in \mathcal{A}$ that the set $b^{-1}(\{x_0\})$ is not compact. In the second case the set $b^{-1}(\{x\})$ is compact for all $x \in \mathcal{A}$ but there exist such compact set $\mathcal{K} \subset \mathcal{A}$ that the set $b^{-1}(\mathcal{K})$ is not compact.[18]

Consider the first case. Since the set $b^{-1}(\{x_0\})$ is not compact it contains sequence $\{\mu_n\}$ which is not relatively compact. By Prohorov's theorem this sequence is not tight [3],[18]. By lemma 1 in [28] and theorem 6.1 in [18] we can consider that this sequence consists of measures with finite support. The below lemma A-1 implies existence of such ε and δ that for any compact set $\mathcal{K} \subseteq \mathcal{A}$ and any natural N there exists $n > N$ such that $\mu_n(U_\delta(\mathcal{K})) < 1 - \varepsilon$. Let $\{\mathcal{K}_n\}$ be increasing sequence of compact convex subsets of $\mathcal{A}$ such that $\bigcup_{n\in\mathbb{N}} U_{\delta/2}(\mathcal{K}_n) \supseteq \mathcal{A}$. For each n let

$$f_n(x) = 1 - 2\delta^{-1} \inf_{y \in U_{\delta/2}(\mathcal{K}_n)} \|x - y\|, \quad x \in \mathcal{A}. \tag{29}$$

Thus f_n is a concave continuous bounded function on the set $\mathcal{A}$ for each n such that $f_n(x) = 1$, $x \in U_{\delta/2}(\mathcal{K}_n)$, and $f_n(x) < 0$, $x \in \mathcal{A} \setminus U_\delta(\mathcal{K}_n)$.

[18] As an example of a convex set corresponding to this case one can consider the set $\mathfrak{T}_1(\mathcal{H})$ endowed with the $\|\cdot\|_p$-norm topology for $p > 1$.

It is clear that $f_0(x)=\lim_{n\to+\infty} f_n(x)\equiv 1$ so that $\overline{\mathrm{co}}f_0(x)\equiv 1$ while the property of the sequence $\{\mu_n\}$ implies that for each n there exists n' such that $\mu_{n'}(U_\delta(\mathcal{K}_n))<1-\varepsilon$ and hence

$$\mathrm{co}f_n(x_0)\le\int_{\mathcal{A}} f_n(x)\mu_{n'}(dx)<1-\varepsilon.$$

Consider the second case. Since the set $b^{-1}(\mathcal{K})$ is not compact it contains sequence $\{\mu_k\}$ which is not relatively compact and such that the sequence $\{x_k=b(\mu_k)\}$ is converging. Similarly to the first case we can consider that this sequence consists of measures with finite support and one can find such ε and δ that for any compact set $\mathcal{K}\subseteq\mathcal{A}$ and any natural N there exists $k>N$ such that $\mu_k(U_\delta(\mathcal{K}))<1-\varepsilon$. Let x_0 be the limit of the sequence $\{x_k\}$. By Prohorov's theorem the compact set $b^{-1}(\{x_0\})$ is tight. Hence there exists such compact set $\mathcal{K}_0\subseteq\mathcal{A}$ that $\mu(\mathcal{K}_0)\ge 1-\varepsilon/2$ for all $\mu\in b^{-1}(\{x_0\})$.

Let $\{\mathcal{K}_n\}$ be increasing sequence of compact convex subsets of $\mathcal{A}$ such that $\mathcal{K}_0\subseteq\mathcal{K}_n$ for all n and $\bigcup_{n\in\mathbb{N}} U_{\delta/2}(\mathcal{K}_n)\supseteq\mathcal{A}$. We will show that for the function f_n defined in (29) the following inequality holds

$$\overline{\mathrm{co}}f_n(x_0)<1-\varepsilon,\quad\forall n. \tag{30}$$

Indeed, the property of the sequence $\{\mu_k\}$ implies that for each n and N there exists $k>N$ such that $\mu_k(U_\delta(\mathcal{K}_n))<1-\varepsilon$ and by noting that μ_k is a measure with finite support we obtain

$$\overline{\mathrm{co}}f_n(x_k)\le\mathrm{co}f_n(x_k)\le\int_{\mathcal{A}} f_n(x)\mu_k(dx)<1-\varepsilon.$$

This and lower semicontinuity of the function $\overline{\mathrm{co}}f_n$ imply (30). $\square$

Lemma A-1. *The subset $\mathcal{P}_0\subseteq\mathcal{P}(\mathcal{A})$ is tight if and only if for all $\varepsilon>0$ and $\delta>0$ there exists compact subset $\mathcal{K}(\varepsilon,\delta)\subseteq\mathcal{A}$ such that*

$$\mu(U_\delta(\mathcal{K}(\varepsilon,\delta)))\ge 1-\varepsilon$$

for all $\mu\in\mathcal{P}_0$, where $U_\delta(\mathcal{K}(\varepsilon,\delta))$ is the closed δ-vicinity of the set $\mathcal{K}(\varepsilon,\delta)$.

Proof. It is easy to see that tightness of the set $\mathcal{P}_0$ implies validity of the condition in the lemma. Suppose this condition holds. For arbitrary $\varepsilon>0$ and each $n\in\mathbb{N}$ let $\mathcal{K}_n=\mathcal{K}\left(\varepsilon 2^{-n},\varepsilon 2^{-n}\right)$. Then for the compact set $\mathcal{K}=\bigcap_{n\in\mathbb{N}} U_{\varepsilon 2^{-n}}(\mathcal{K}_n)$ we have

$$\mu(\mathcal{A}\setminus\mathcal{K})\le\sum_{n=1}^{+\infty}\mu(\mathcal{A}\setminus U_{\varepsilon 2^{-n}}(\mathcal{K}_n))<\sum_{n=1}^{+\infty}\varepsilon 2^{-n}<\varepsilon$$

for all $\mu\in\mathcal{P}_0$, which means that the set $\mathcal{P}_0$ is tight. $\square$

7.3 One property of posteriori states

Let $\mathfrak{M}$ be an arbitrary instrument on the set $\mathfrak{S}(\mathcal{H})$ with the set of outcomes $\mathcal{X}$ [9]. For a given arbitrary state $\rho \in \mathfrak{S}(\mathcal{H})$ let $\mu_\rho(\cdot) = \mathrm{Tr}\mathfrak{M}(\cdot)(\rho)$ be the posteriori measure on the set $\mathcal{X}$ and $\{\sigma(x|\rho)\}_{x\in\mathcal{X}}$ be the family of a posteriori states corresponding to the a priori state ρ [16].

Lemma A-2. *For arbitrary convex lower semicontinuous function f on the set $\mathfrak{S}(\mathcal{H})$ and arbitrary sequence $\{\rho_n\} \subset \mathfrak{S}(\mathcal{H})$ converging to a state ρ_0 the following relation holds*

$$\liminf_{n\to+\infty} \int_{\mathcal{X}} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \geq \int_{\mathcal{X}} f(\sigma(x|\rho_0))\mu_{\rho_0}(dx).$$

Proof. By lemma 2 we can consider that the function f is lower bounded. It is sufficient to show that the assumption

$$\lim_{n\to+\infty} \int_{\mathcal{X}} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \leq \int_{\mathcal{X}} f(\sigma(x|\rho_0))\mu_{\rho_0}(dx) - \Delta, \quad \Delta > 0, \tag{31}$$

leads to a contradiction.

Let $\nu_0 = \mu_{\rho_0} \circ \sigma^{-1}(\,\cdot\,|\rho_0)$ be the image of the measure μ_{ρ_0} under the map $x \mapsto \sigma(x|\rho_0)$. It is clear that $\nu_0 \in \mathcal{P}$ (see remark 8) and that

$$\int_{\mathcal{X}} f(\sigma(x|\rho_0))\mu_{\rho_0}(dx) = \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_0(d\rho).$$

By separability of the set $\mathfrak{S}(\mathcal{H})$ for given m one can find family $\{\mathcal{B}_i^m\}_i$ of Borel subsets of $\mathfrak{S}(\mathcal{H})$ such that $\nu_0(\mathcal{B}_i^m) > 0$ for all i and the sequence of measures

$$\nu_m = \left\{ \nu_0(\mathcal{B}_i^m), \frac{1}{\nu_0(\mathcal{B}_i^m)} \int_{\mathcal{B}_i^m} \rho\nu_0(d\rho) \right\}$$

weakly converges to the measure ν_0 (see the proof of lemma 1 in [11]). Lower semicontinuity of the function $\mu \mapsto \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\mu(d\rho)$ implies existence of such m_0 that

$$\begin{gathered}\sum_i \nu_0(\mathcal{B}_i^{m_0}) f\left(\frac{1}{\nu_0(\mathcal{B}_i^{m_0})} \int_{\mathcal{B}_i^{m_0}} \rho\nu_0(d\rho) \right) = \\ \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_{m_0}(d\rho) \geq \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_0(d\rho) - \tfrac{1}{3}\Delta.\end{gathered} \tag{32}$$

By using the finite family $\{\mathcal{X}_i = \sigma^{-1}(\mathcal{B}_i^{m_0}|\rho_0)\}$ of μ_{ρ_0}-measurable subsets of $\mathcal{X}$ we can construct the family $\{\mathcal{X}_i'\}$ consisting of the same number of Borel subsets of $\mathcal{X}$ such that $\mu_{\rho_0}((\mathcal{X}_i' \setminus \mathcal{X}_i) \cup (\mathcal{X}_i \setminus \mathcal{X}_i') = 0$ and $\bigcup_i \mathcal{X}_i = \mathcal{X}$. For each i the state

$$\sigma_0^i = \frac{1}{\nu_0(\mathcal{B}_i^{m_0})} \int_{\mathcal{B}_i^{m_0}} \rho\nu_0(d\rho) = \frac{1}{\mu_{\rho_0}(\mathcal{X}_i')} \int_{\mathcal{X}_i'} \sigma(x|\rho_0)\mu_{\rho_0}(dx) = \frac{\mathfrak{M}(\mathcal{X}_i')(\rho_0)}{\mathrm{Tr}\mathfrak{M}(\mathcal{X}_i')(\rho_0)}$$

is the posteriori state, corresponding to the set $\mathcal{X}_i'$ of outcomes and the a priori state ρ_0.

For each i let $\sigma_n^i = \dfrac{\mathfrak{M}(\mathcal{X}_i')(\rho_n)}{\mathrm{Tr}\mathfrak{M}(\mathcal{X}_i')(\rho_n)}$ be the posteriori state, corresponding to the set $\mathcal{X}_i'$ of outcomes and the a priori state ρ_n.[19] By lower semicontinuity of the function f and since $\lim_{n\to+\infty} \mathfrak{M}(\mathcal{X}_i')(\rho_n) = \mathfrak{M}(\mathcal{X}_i')(\rho_0)$ we have

$$\sum_i \mu_{\rho_n}(\mathcal{X}_i')f(\sigma_n^i) \geq \sum_i \mu_{\rho_0}(\mathcal{X}_i')f(\sigma_0^i) - \tfrac{1}{3}\Delta \tag{33}$$

for all sufficiently large n.

By Yensen's inequality (proposition A-2) convexity and lower semicontinuity of the function f implies

$$\mu_{\rho_n}(\mathcal{X}_i')f(\sigma_n^i) \leq \int_{\mathcal{X}_i'} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx), \quad \forall i, n. \tag{34}$$

By using (32),(33) and (34) we obtain

$$\int_{\mathcal{X}} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) = \sum_i \int_{\mathcal{X}_i'} f(\sigma(x|\rho_n))\mu_{\rho_n}(dx) \geq \sum_i \mu_{\rho_n}(\mathcal{X}_i')f(\sigma_n^i)$$

$$\geq \sum_i \mu_{\rho_0}(\mathcal{X}_i')f(\sigma_0^i) - \tfrac{1}{3}\Delta \geq \int_{\mathfrak{S}(\mathcal{H})} f(\rho)\nu_0(d\rho) - \tfrac{2}{3}\Delta$$

for all sufficiently large n, which contradicts to (31) $\square$.

The author is grateful to A.S.Holevo for the help and useful discussion.

[19] Since $\mathrm{Tr}\mathfrak{M}(\mathcal{X}_i')(\rho_0) = \mu_{\rho_0}(\mathcal{X}_i') > 0$ the state σ_n^i is correctly defined for all sufficiently large n.

References

[1] E.Alfsen "Compact convex sets and boundary integrals", Springer Verlag, 1971;

[2] C.H.Bennett, D.P.DiVincenzo, J.A.Smolin, W.K.Wootters "Mixed State Entanglement and Quantum Error Correction", Phys. Rev. A 54, 3824-3851, 1996, arXiv:quant-ph/9604024;

[3] P.Billingsley "Convergence of probability measures", John Willey and Sons. Inc., New York-London-Sydney-Toronto, 1968;

[4] R. O'Brien "On the openness of the barycentre map", Math. Ann. V.223, N.3, P.207-212, 1976;

[5] A.Clausing, S.Papadopoulou "Stable convex sets and extremal operators", Math.Ann. V.231, P.193-203, 1978;

[6] J.Eisert, C.Simon, M.B.Plenio "The quantification of entanglement in infinite-dimensional quantum systems", J. Phys. A 35, 3911, 2002, arXiv:quant-ph/0112064;

[7] J.Eisert "Entanglement in quantum information theory", arXiv:quant-ph/0610253;

[8] R.Grzaslewicz "Extreme continuous function property", Acta.Math.Hungar. V.74, P.93-99, 1997;

[9] A.S.Holevo "Statistical structure of quantum theory", Springer-Verlag, 2001;

[10] A.S.Holevo "Classical capacities of quantum channels with constrained inputs", Probability Theory and Applications, 48, N.2, 359-374, 2003, arXiv:quant-ph/0211170;

[11] A.S.Holevo, M.E.Shirokov "Continuous ensembles and the χ-capacity of infinite dimensional channels", Probability Theory and Applications, 50, N.1, 98-114, 2005, arXiv:quant-ph/0408176;

[12] A.S.Holevo, M.E.Shirokov, R.F.Werner "On the notion of entanglement in Hilbert space", Russian Math. Surveys, 60, N.2, 153-154, 2005, arXiv:quant-ph/0504204;

[13] A.Lima, "On continuous convex functions and split faces", Proc. London Math. Soc., 25, N.3, 27–40, 1972.

[14] A.D.Joffe, W.M.Tikhomirov "Theory of extremum problems", AP, NY, 1979;

[15] T.J. Osborne "Convex Hulls of Varieties and Entanglement Measures Based on the Roof Construction", Quantum Information and Computation, V.7, N.3, P.209-227, 2007, arXiv:quant-ph/0402055v2;

[16] M.Ozawa "Conditional probability and posteriori states in quantum mechanics", Publ. Res. Inst. Math. Sci. V.21, N.2, P.279-295, 1985;

[17] S.Papadopoulou "On the geometry of stable compact convex sets", Math.Ann. V.229, P.193-200, 1977;

[18] K.Parthasarathy "Probability measures on metric spaces", Academic Press, New York and London, 1967;

[19] M.B.Plenio, S.Virmani "An introduction to entanglement measures", quant-ph/0504163;

[20] V.Yu.Protasov, M.E.Shirokov, "Generalized compactness in linear spaces and its applications", Sbornik: Mathematics V.200, N.5 (expected), 2009;

[21] P.Ressel "Some continuiuty and measurability results on spaces of measures", Math. Scand., V.40, N.1, P.69-78, 1977;

[22] R.Rockafellar "Convex analysis", Tyrrell, 1970;

[23] P.Rungta, C.M.Caves "Concurrence-based entanglement measures for isotropic states", Phys. Rev. A, V.67, N.1, P.012307/1-9, 2003;

[24] M.E.Shirokov "The Holevo capacity of infinite dimensional channels and the additivity problem", Comm. Math. Phys., V.262, P.137-159, 2006, arXiv:quant-ph/0408009;

[25] M.E.Shirokov "On entropic quantities related to the classical capacity of infinite dimensional quantum channels", arXiv:quant-ph/0411091, 2004;

[26] M.E.Shirokov "Entropic characteristics of subsets of states", arXiv:quant-ph/0510073, 2005;

[27] M.E.Shirokov "Properties of probability measures on the set of quantum states and their applications", arXiv:math-ph/0607019, 2006;

[28] M.E.Shirokov "On the strong CE-property of convex sets", Mathematical Notes, V.82, N.3, P.395-409, 2007;

[29] M.E.Shirokov "The continuity condition for the von Neumann entropy based on the special approximation", to appear in arXiv:quant-ph;

[30] A.Uhlmann "Entropy and optimal decomposition of states relative to a maximal commutative subalgebra", arXiv:quant-ph/9704017;

[31] N.N.Vahania, V.I.Tarieladze "Covariant operators of probability measures in locally convex spaces", Probability Theory and Applications, V.23, N.1, P.1-23, 1978;

[32] G.Vidal "Entanglement monotones", J.Mod.Opt. V.47, P.355, 2000, arXiv:quant-ph/9807077;

[33] A.Wehrl "General properties of entropy", Rev. Mod. Phys. V.50, P.221-250, 1978;

[34] K.Zyczkowski, I.Bengtsson "An Introduction to Quantum Entanglement: A Geometric Approach", 2006, arXiv:quant-ph/0606228.